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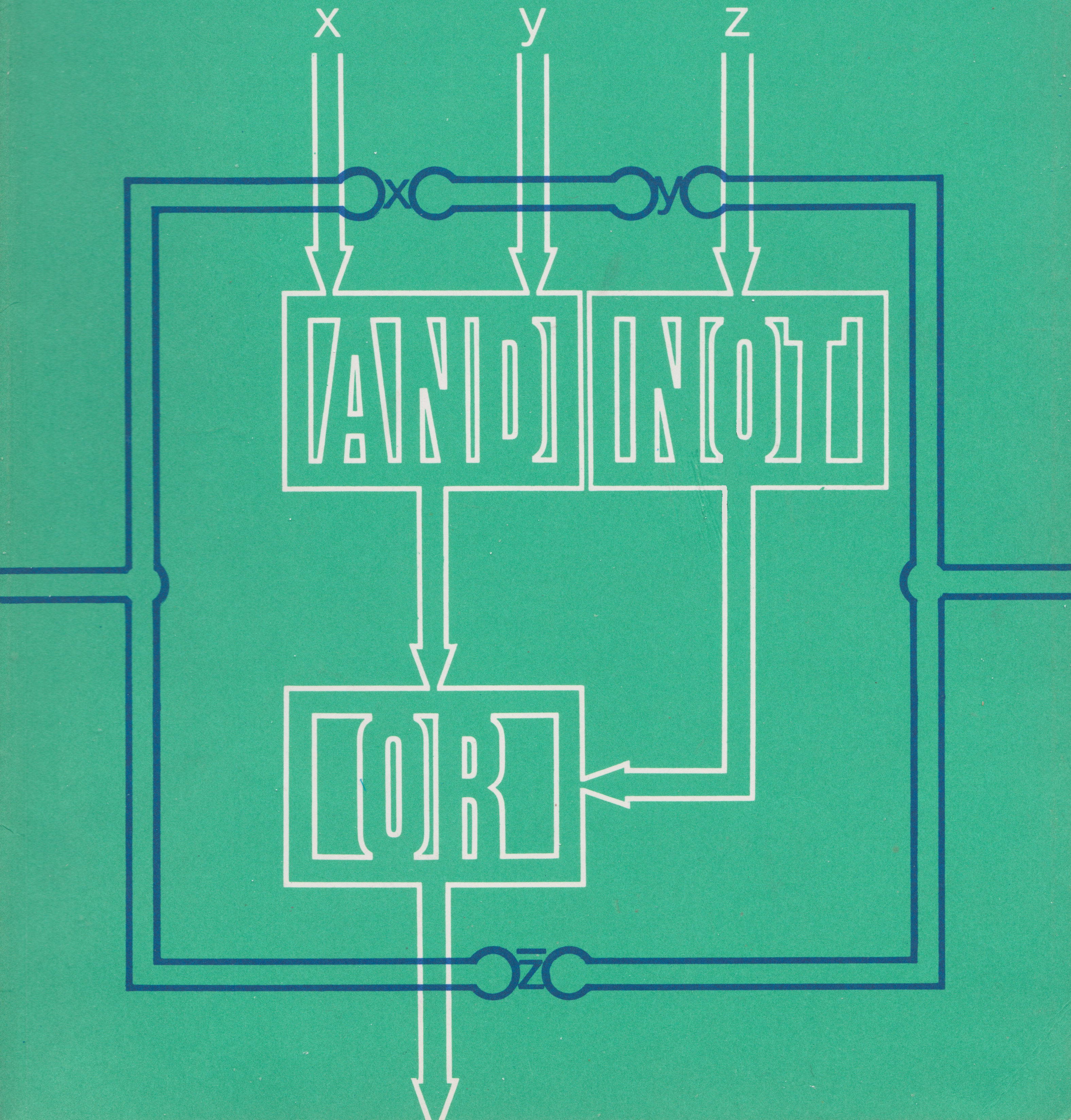
Mathematics Foundation Course Unit 11

Logic (I) Boolean Algebra



The Open University

Mathematics Foundation Course Unit 11

LOGIC I—BOOLEAN ALGEBRA

Prepared by the Mathematics Foundation Course Team

Correspondence Text 11

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Contents

Objectives

The general aim of this unit is to show that set theory, logic and the behaviour of certain switching circuits all have the same algebraic structure, which we call Boolean Algebra. This enables expressions in any of these subjects to be simplified by algebraic and other techniques so that results obtained in one of them may be interpreted in the others. After working through this unit, you should be able to:

(i) recognize the names and symbols of the principal logical connectives; explain these in words and deduce the corresponding truth tables;

(ii) establish whether the logical connectives, regarded as binary operations on the set of all propositions, have the commutative, associative and distributive properties;

(iii) justify the proposition that certain sets of connectives are adequate for the generation of all connectives and that certain connectives may be regarded as primitive in this context;

(iv) represent by symbols compound statements which are given in words, and hence establish their truth values by the use of truth tables;

(v) determine whether a given logical expression represents a tautology, a contradiction, or neither;

(vi) manipulate Boolean expressions algebraically and interpret the result in terms of: sets; propositions; switching circuits;

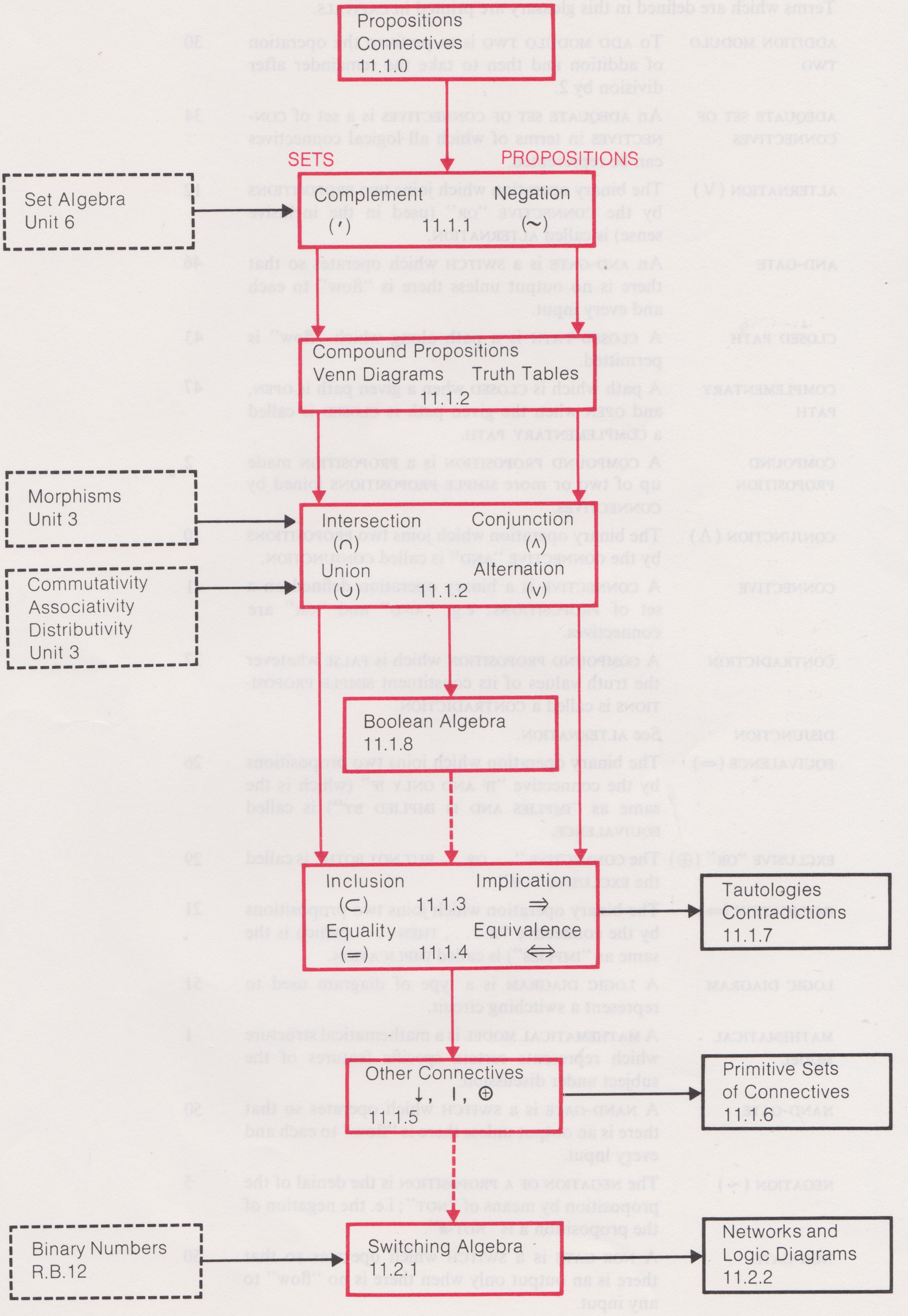
(vii) given a switching network or logic diagram, deduce the algebraic expression corresponding to it, and vice versa;

(viii) design a simple switching network, given the appropriate specifications.

Note

Before working through this correspondence text, make sure you have read the general introduction to the mathematics course in the Study Guide, as this explains the philosophy underlying the whole course. You should also be familiar with the section which explains how a text is constructed and the meanings attached to the stars and other symbols in the margin, as this will help you to find your way through the text.

Structural Diagram



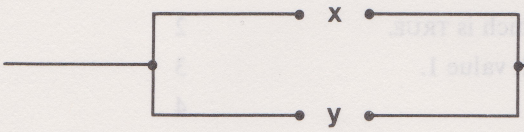
Glossary

Page

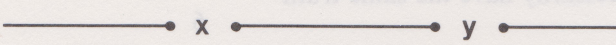
Terms which are defined in this glossary are printed in CAPITALS.

ADDITION MODULO TWO	To ADD MODULO TWO is to perform the operation of addition and then to take the remainder after division by 2.	30
ADEQUATE SET OF CONNECTIVES	An ADEQUATE SET OF CONNECTIVES is a set of CONNECTIVES in terms of which all logical connectives can be represented.	34
ALTERNATION ($\vee$)	The binary operation which joins two PROPOSITIONS by the CONNECTIVE "OR" (used in the inclusive sense) is called ALTERNATION.	12
AND-GATE	An AND-GATE is a SWITCH which operates so that there is no output unless there is "flow" to each and every input.	46
CLOSED PATH	A CLOSED PATH is a path along which "flow" is permitted.	43
COMPLEMENTARY PATH	A path which is CLOSED when a given path is OPEN, and OPEN when the given path is CLOSED is called a COMPLEMENTARY PATH.	47
COMPOUND PROPOSITION	A COMPOUND PROPOSITION is a PROPOSITION made up of two or more SIMPLE PROPOSITIONS joined by CONNECTIVES.	2
CONJUNCTION ($\wedge$)	The binary operation which joins two PROPOSITIONS by the CONNECTIVE "AND" is called CONJUNCTION.	10
CONNECTIVE	A CONNECTIVE is a binary operation defined on a set of PROPOSITIONS; e.g. "AND" and "OR" are connectives.	1
CONTRADICTION	A COMPOUND PROPOSITION which is FALSE whatever the truth values of its constituent SIMPLE PROPOSITIONS is called a CONTRADICTION.	37
DISJUNCTION	See ALTERNATION.	
EQUIVALENCE ($\Leftrightarrow$)	The binary operation which joins two propositions by the connective "IF AND ONLY IF" (which is the same as "IMPLIES AND IS IMPLIED BY") is called EQUIVALENCE.	26
EXCLUSIVE "OR" ($\oplus$)	The CONNECTIVE ". . . OR . . . BUT NOT BOTH" is called the EXCLUSIVE "OR".	29
IMPLICATION ($\Rightarrow$)	The binary operation which joins two propositions by the connective "IF . . . , THEN . . ." (which is the same as "IMPLIES") is called IMPLICATION.	21
LOGIC DIAGRAM	A LOGIC DIAGRAM is a type of diagram used to represent a switching circuit.	51
MATHEMATICAL MODEL	A MATHEMATICAL MODEL is a mathematical structure which represents certain specific features of the subject under discussion.	1
NAND-GATE	A NAND-GATE is a SWITCH which operates so that there is an output unless there is "flow" to each and every input.	50
NEGATION ($\sim$)	The NEGATION OF A PROPOSITION is the denial of the proposition by means of "NOT"; i.e. the negation of the proposition **a** is "NOT-**a**".	5
NOR-GATE	A NOR-GATE is a SWITCH which operates so that there is an output only when there is no "flow" to any input.	50

		Page
NOT-GATE	A NOT-GATE is a SWITCH whose output is the complement of its input.	53
OPEN PATH	An OPEN PATH is a path along which "flow" is prevented.	43
OR-GATE	An OR-GATE is a SWITCH which operates so that there is an output when there is "flow" to any input.	46
PARALLEL CONNECTION	A PARALLEL CONNECTION is a connection of two paths side by side so as to provide alternative paths between two points, e.g. for paths x, y:	44



PRIMITIVE CONNECTIVE	A PRIMITIVE CONNECTIVE is a CONNECTIVE in terms of which all logical connectives can be represented.	35
PROPOSITION	A PROPOSITION is a statement which must be *either* TRUE *or* FALSE.	2
SERIES CONNECTION	A SERIES CONNECTION is a connection of two paths one after the other so that both must be traversed, e.g. for paths x, y:	44



SIMPLE PROPOSITION	A SIMPLE PROPOSITION is a PROPOSITION which cannot be broken down into other propositions joined by one or more CONNECTIVES.	2
SWITCH	A SWITCH is a two-state device by means of which a path is made *either* OPEN *or* CLOSED.	43
TAUTOLOGY	A TAUTOLOGY is a COMPOUND PROPOSITION which is true whatever the truth values of its constituent SIMPLE PROPOSITIONS.	3, 36
TRUTH TABLE	A TRUTH TABLE is a table showing the TRUTH VALUE of a COMPOUND PROPOSITION corresponding to the separate truth values of its constituent SIMPLE PROPOSITIONS.	6
TRUTH VALUE	The TRUTH VALUE of a PROPOSITION **a** is 1 if **a** is TRUE and 0 if **a** is FALSE.	2
UNIVERSE (or UNIVERSAL SET)	The UNIVERSE is the set of all elements under consideration.	4
VENN DIAGRAM	A VENN DIAGRAM is a diagram in which sets are represented by regions of a plane, and on which complements, unions and intersections of sets may be indicated.	3

Notation

The symbols are presented in the order in which they appear in the text.

		Page
$x \in A$	The element x is a member of the set A.	1
$\mathbf{a}$	The proposition $\mathbf{a}$.	2
P	The set of all propositions.	2
T	The mapping which assigns a truth value to each element of P.	2
0	The truth value of a proposition which is FALSE.	2
1	The truth value of a proposition which is TRUE.	2
$\mathbf{a} \longmapsto 1$	The proposition $\mathbf{a}$ maps to the truth value 1.	3
U	The universal set.	4
Z	The set of all integers.	4
R	The set of all real numbers.	4
A'	The complement of the set A with respect to the universal set U.	4
$x \notin A$	The element x is not a member of the set A.	5
$\sim$	The symbol for negation (read as "not").	5
Z^+	The set of all positive integers.	6
$\mathbf{a} = \mathbf{b}$	The propositions $\mathbf{a}$ and $\mathbf{b}$ necessarily have the same truth values (read as "$\mathbf{a}$ equals $\mathbf{b}$").	6
$A \cap B$	The set of all elements which belong *both* to set A and to set B.	6
$\wedge$	The symbol for conjunction (read as "and").	7
$\square$	A binary operation on a set (read as "square").	10
$(\mathbf{a}, \mathbf{b})$	The ordered pair of propositions $\mathbf{a}$, $\mathbf{b}$.	11
$A \cup B$	The set of all elements which belong to set A *or* to set B *or* to both.	11
$\vee$	The symbol for alternation (read as "or").	11
$\Rightarrow$	The symbol for implication (read as "implies").	17
$A \subseteq B$	The set A is a subset of the set B.	17
$\Leftrightarrow$	The symbol for equivalence (read as "if and only if").	23
$\downarrow$	The symbol for the negation of alternation ($\mathbf{a} \downarrow \mathbf{b}$ is read as "*neither* $\mathbf{a}$ *nor* $\mathbf{b}$").	28
$\mid$	The symbol for the negation of conjunction ($\mathbf{a} \mid \mathbf{b}$ is read as "*either* not-$\mathbf{a}$ *or* not-$\mathbf{b}$").	29
$\oplus$	The symbol for the "exclusive or". ($\mathbf{a} \oplus \mathbf{b}$ is read as "*either* $\mathbf{a}$ *or* $\mathbf{b}$ *but not both*").	29
$\oplus_2$	The operation "add and then take the remainder on division by 2".	30
$\cap$, $\cup$	Binary operations in a Boolean Algebra (read as "cap" and "cup" respectively).	39
O, I	The "identity" elements in a Boolean Algebra.	40
$\varnothing$	The empty set.	41
$\mathbf{F}$	"Universal falsehood".	41
$\mathbf{T}$	"Universal truth".	41
x	The switch x.	43
$\bar{x}$	The switch which is complementary to x.	47

Note

In this unit we use bold-face type to denote propositions (see the symbol **a** above). When writing in ordinary longhand, it is not practical to try to imitate bold-face printing, so conventionally we place a "twiddle" underneath a letter which corresponds to a bold-face letter in print. You should thus represent the bold-face letters **a**, **b**, **c**, . . . when you write them by

$$\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}, \ldots$$

(The "twiddle" is in fact the normal way of conveying to a printer that bold-face type should be used.)

Bibliography

C. B. Allendoerfer and C. O. Oakley, *Principles of Mathematics*, 2nd ed. (McGraw-Hill, 1963).
The basic logic connectives are discussed in Chapter 1, where the link with set algebra is also established. The material is developed further in Chapter 15, which concludes with some simple ideas on switching.

H. G. Flegg, *Boolean Algebra* (John Wiley, 1964) (Transworld: abridged edition, 1972).
This book discusses Boolean Algebra with particular emphasis on its principal applications, and covers most of the material of the unit.

B. Arnold, *Logic and Boolean Algebra* (Prentice-Hall, 1962).
This book concentrates essentially on the algebraic structure of Boolean algebras and Boolean rings, and is particularly suitable for students who would like a more formal presentation of the algebra itself than that which is given in the unit.

Note

For those with a historical bent, George Boole's original work, *An Investigation of the Laws of Thought*, first published in 1854, has been reprinted as a Dover paperback (1958) and is currently available.

11.0 INTRODUCTION

Introduction
* *

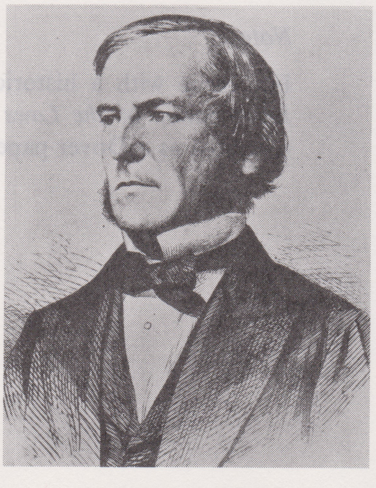
There is a story of a man out for a walk who finds himself approaching an unsignposted T-junction situated in the middle of a remote village.

Now all the inhabitants of this village have a curious habit: any inhabitant *either* invariably tells the truth *or* invariably tells a lie. The traveller wants to know which way to turn at the T-junction in order to reach a given destination, X (say), but he is allowed to ask only one question and to ask it of only one of the village's inhabitants, who may reply *either* "Yes" *or* "No". The man must solve the problem: Is there a question he can ask in order to be certain that the answer given will reveal to him the correct route, and, if so, what is a suitable question?

Now, of course, this is a purely hypothetical problem, entertaining though it may be. Solving it, however, is a legitimate logical exercise. At first sight, the solution may seem very difficult to obtain; you may even think that there is no suitable question. In fact, given facility with the kind of logical procedures which we discuss in this unit, the solution of the traveller's problem is obtained quickly and simply: we shall demonstrate this as soon as the necessary ideas have been discussed.

However, there are many less hypothetical problems, which may be investigated by the elementary logical procedures described in this unit. There are a whole host of problems relating to safety precautions in factories, in travel, etc., which are fundamentally problems in logic. For example, a machine operator must not be able to activate a certain potentially dangerous machine unless a series of precautionary operations have been carried out first; a railway signal must not give an oncoming train the green light if points are incorrectly switched or if there is another train on the section of line ahead.

Any system whose basic elements are of the ON/OFF type (that is, two-state devices) can be regarded as a logic system; the material of this unit provides a theoretical foundation by means of which such systems may be analysed or designed. One of the most important modern developments using principles of logic is the digital computer. Most, though not all, digital computers work in binary arithmetic using only the digits 0 and 1; however, all are basically composed of two-state devices. Logic plays a fundamental role in the theory of two-state devices because, having stipulated that a given proposition (i.e. a precisely defined statement) in logic must be *either* TRUE *or* FALSE, we find ourselves in just such a two-state situation as exists in the case of a switch, which must be *either* ON *or* OFF, or of a digit place in a binary number, which must be *either* 0 *or* 1.



George Boole 1815–1864
(*Science Museum Library*)

This unit on Logic is sub-titled "Boolean Algebra" because the system of logic to which we shall confine ourselves can be completely described in terms of a special algebra which has been developed from the work of the nineteenth century philosopher-mathematician, George Boole. We shall prove a number of theorems in this algebra, but we shall not consider methods of proof as such, nor shall we attempt to philosophize on the relation between proof and truth. These matters will be discussed in *Unit 17, Logic II — Proof*.

11.1 SETS AND PROPOSITIONS

11.1

11.1.0 Introduction

11.1.0 Introduction
* *

In section 11.1 we shall develop the algebra known as Boolean Algebra. We shall see how this algebra arises naturally from a consideration of sets, and that it gives a *model* of an elementary system of logic.

You have enountered the concept of a set in most of the earlier units of this course. In particular, in *Unit 6, Inequalities* we discussed the idea of a set and its complement, and also the set operations of union and intersection. You will therefore find that a quick revision of the last part of section 6.2.3 will prove helpful in studying the material which we shall develop here.

The system of logic considered in this unit is the one known as the propositional calculus. This is concerned with the way in which we can build up compound statements by combining short simple statements like

Definitions and Notation
* * *

"it is raining",

"$x \in A$",

"$y < z$",

using linking words, known as connectives, such as

Definition 1
* * *

"and",

"or",

"if . . . , then . . .".

Thus,

"if it is raining, then I shall get wet"

is a *compound statement* made up of the *simple statements*:

"it is raining"

and

"I shall get wet"

linked together by the connective

"if . . . , then . . ."

(We have chosen to use the word "statement" rather than the more usual word "sentence", because we wish to exclude sentences which are questions, commands, etc.)

The propositional calculus tells us how the truth or falsehood of a compound statement is determined by the truth or falsehood of the constituent simple statements of which it is formed. Because we have already introduced set operations in *Unit 6* we shall consider the *algebra of sets* and the *propositional calculus* side by side, thus presenting the new ideas of this unit alongside ideas with which you are already familiar.

We take ordinary statements in English as our starting point for the propositional calculus, but we must remember that everyday English is an *informal* language; that is, it is a language whose grammar is subject to modification of style, and the way in which some given truth is correctly expressed in the language may be largely a matter of opinion. You will be only too well aware of the ambiguities which can arise in ordinary conversation and of the ambiguous language in which many political speeches are deliberately phrased. If we are to develop a "calculus" of statements, we need to have a firmer foundation than ordinary speech or writing can provide. We need to study what we call a *formal* language in which ambiguities of style and phrasing are avoided, and where every

statement is *either* TRUE *or* FALSE (but not both); any statement having this property is called a proposition. In the propositional calculus we restrict ourselves not only to sentences which are statements, but also to those statements which are propositions.

Definition 2

In order to illustrate the need for careful definitions of terms, consider the proposition:

"the temperature is low".

Before we can determine whether this proposition is TRUE or FALSE, we must make the meaning of "low" precise; for example, by defining "low" to mean "less than 5 °C recorded on a given thermometer in a given situation". If we do not have precise definitions for all our words and phrases, then we shall have only an informal language which may give rise to ambiguities. From now on we shall assume that all the statements under consideration are propositions, and we shall speak of simple and compound propositions rather than of simple and compound statements.

Consider the proposition:

$x \in A$.

We note that this is a simple proposition because it cannot be broken down into constituent parts which are also propositions. By contrast, the proposition:

Definition 3

$x \in A$ **and** $y \in B$,

is a compound proposition which can be broken down into constituent parts which are themselves propositions and which are joined together by the connective "and".

Definition 4

We begin by abstracting certain properties from our informal language. In particular, we abstract the logical properties of the connectives with which we combine simple propositions to form compound propositions. We have to make precise rules about how these connectives combine propositions, and, in order to build up an algebra, we need to have a symbolic way of representing simple propositions and also the connectives.

We adopt the easiest and most obvious representation of simple propositions by using letters of the alphabet to represent them. Thus, if our simple proposition

"it is raining"

is represented by the letter "**a**", then **a** represents a simple proposition which is *either* TRUE *or* FALSE. We must distinguish between the *proposition* and *the assertion that it is in fact raining* which is equivalent to saying that the proposition "it is raining" is TRUE.

Notation 1

In *Unit 3, Operations and Morphisms* we studied binary operations in some detail, and here we have further examples. We denote by P the set of all propositions:

Definition 5

$P = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots\}$.

The connectives by means of which we build up compound propositions are binary operations on P; when we represent the connectives symbolically we are in the familiar abstract situation of having a set with binary operations defined on it.

We shall also be concerned with the set

$\{0, 1\}$.

In the context of logic, the integers 0 and 1 are called truth values. The reason is that in the propositional calculus each proposition is *either* TRUE *or* FALSE. We can thus define a function, T, from our set of propositions, P, to the set $\{0, 1\}$, whereby a proposition is mapped to 0 if it is

Definition 6

Definition 7

FALSE and to 1 if it is TRUE. For example, if **a** represents "it is raining" and if, for the purpose of a particular investigation, we wish to assert that *it is in fact raining*, then this assertion is equivalent to mapping $\mathbf{a} \longmapsto 1$. That is,

$$T: P \longmapsto \{0, 1\} \quad \text{and} \quad T(\mathbf{a}) = 1.$$

So we have a set of propositions with certain operations defined on it, and we shall map this set to the set $\{0, 1\}$. If we wish the latter set to give us a mathematical model of our set of propositions, we need to define operations on $\{0, 1\}$ which correspond to our proposition connectives, and thus make the function, T, a *morphism*.

(It is important to remember here that in order to be able to assign one of the two truth values to a proposition such as "it is raining", it is first necessary to be precise as to the circumstances under which each truth value would be deemed to hold. In everyday English we usually manage by using common sense. If we say "it is raining" we mean "it is raining here and now" (the "here" meaning "outside" if the conversation is taking place indoors), and we can check the truth of the proposition by trying to see or feel the rain in order to determine if indeed it *is* raining and our proposition is thus TRUE, or if it *is not* raining and our proposition is FALSE.)

Of particular interest is a class of compound propositions whose overall truth values in no way depend upon the truth values of their constituent propositions. If a compound proposition is true, whatever the truth values of its constituent propositions, it is called a tautology. The importance of such propositions lies in the fact that laws of logic and theorems in mathematics are also tautologies, though many of them are of a more complicated kind than those we study in this unit.

Definition 8
* * *

In section 11.1 we shall consider the *algebra of sets* and the *propositional calculus* side by side, and make a special study of certain logical concepts which have particular relevance to mathematicians (and others), such as the concept of a tautology which we have just mentioned. We shall make considerable use of set diagrams* similar to those encountered in *Unit 6, Inequalities*, and also of tables of truth values, and we shall thus develop the algebra which we shall use in the latter part of the unit to study switching systems. In the course of the development of this algebra, you will encounter a number of morphisms, and you should therefore find that your earlier study of *Unit 3, Operations and Morphisms* will greatly help you in appreciating how the algebra can unify the study of sets and the study of propositions and (later) the study of switching systems also.

* Diagrams in which regions of the plane represent sets are called Venn diagrams, after the logician, John Venn. These diagrams provide useful illustrations, but a demonstration by means of a Venn diagram is NOT a proof.

Definition 9
* * *

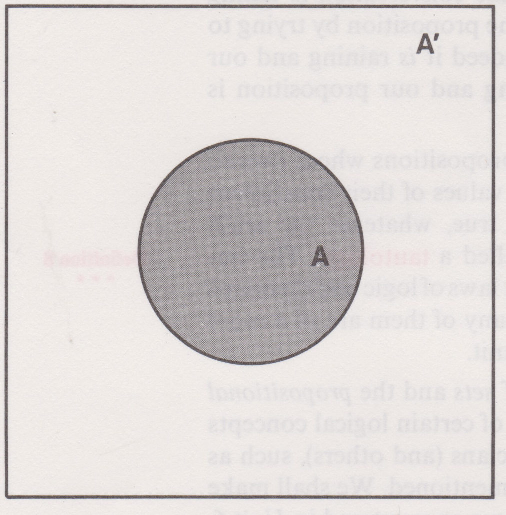
11.1.1 Negation and Complement

11.1.1
Main Text
* * *

In order to introduce the first of the logical concepts, that of a proposition and its negation, we remind you of the essential property of a set, namely that, given any set and any object, we can say whether or not the object is a member of the set. We also introduce the concept of a universe or universal set, which we denote by U. In each discussion about sets there must be a universal set lurking in the background, as otherwise we may get out of our depth and find ourselves in paradoxical situations. So in each discussion about sets we assume that a suitable universal set, U, exists, and that all the sets under consideration are subsets of U. In most cases, we restrict our universe to the particular set of objects which are relevant to whatever problem we have in hand. For example, when discussing sets of integers we take $U = Z$ or $U = R$.

Definition 1
* *

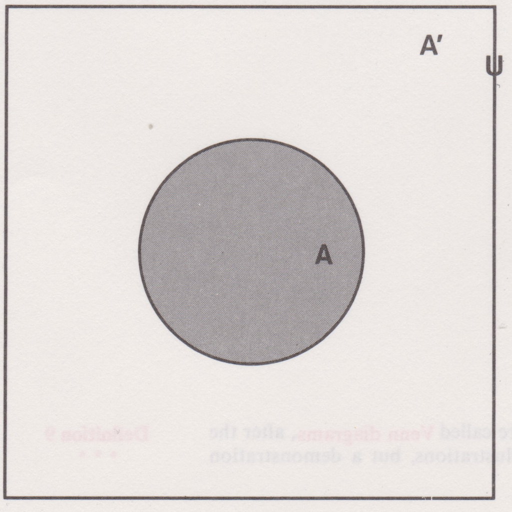
Consider the following Venn diagram:



In this diagram, the universe, U, of all possible objects which we consider, represented by the whole square, is divided into two parts. The shaded area and its boundary represent the set of all the elements which belong to A, and the unshaded area represents the set of all the elements which do not belong to A. This latter set is called the complement of A, and denoted by A'.

Definition 2
* * *

We have already referred to a Venn diagram depicting a set A and its complement, A', with respect to a universal set U:



By specifying a universal set U together with a set A (which must be a subset of U), we have also completely determined the complement of A with respect to U. Thus, if

$$U = \{2, 3, 4, 5, 6, 7\}$$

and

$$A = \{3, 4, 5, 7\},$$

then the complement, A', is the set $\{2, 6\}$. Notice that the complement of the complement of A, that is, $(A')'$, is the set A itself. Thus, within a given universal set, subsets occur naturally in pairs, each member of such a pair being the complement of the other.

As with sets, propositions also arise in pairs. The proposition

$$x \in A$$

occurs naturally with the proposition

$$x \notin A$$

(meaning "x is not a member of A" or "x does not belong to A"), which, provided that we confine ourselves to elements x of some specified universal set, has the alternative representation,

$$x \in A'.$$

Notice particularly that, given two propositions, such as

$$x \in A$$

and

$$x \notin A,$$

one of the two must be TRUE and the other FALSE, this being a defining property of propositions. Each of these propositions is said to be the negation of the other.

Definition 3

The logic symbol representing negation is $\sim$ and hence the negation of any proposition **a** is represented by

Notation 1

$$\sim \mathbf{a}$$

which we read as "not-a".

Thus, if **a** stands for

$$x \in A,$$

then $\sim$**a** represents

$$x \notin A$$

or

$$x \in A'.$$

(Complementation and negation behave similarly, as we shall see, but you should be careful to distinguish between them, in particular by keeping in mind the sets on which they are defined.)

We can think of negation as a *unary operation* (or function) on the set of all propositions, P. (We discussed unary operations in *Unit 3*.) This operation is not a connective since it does not combine propositions. However, if we allocate a truth value from the set $\{0, 1\}$ to **a**, we can write down a truth value for the negation of **a**, and thus form a table:

a	$\sim$**a**
0	1
1	0

Such a table is called a truth table. Of course, this particular truth table is a very simple one and you may feel that it is hardly worth writing down. However, we shall see that the truth table concept, though simple in essence, is very useful in more complicated situations.

Definition 4
* * *

Exercise 1

Exercise 1

(i) Write down a negation of each of the following propositions: (2 minutes)
  (a) The man has blue eyes.
  (b) $x \leqslant y$, where x and y are real numbers.
  (c) The temperature is below 10 °C.
  (d) It is not foggy.

(ii) In mathematics we use "=" in many situations with many different meanings. Can you say in words how we should define equality of propositions, that is, the symbol "=" in $\mathbf{a} = \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are propositions? Can you express your answer in terms of the truth values of $\mathbf{a}$ and $\mathbf{b}$? (2 minutes)

(iii) Show by means of a truth table that (2 minutes)

$$\sim \sim \mathbf{a} = \mathbf{a}$$

and

$$\sim \sim \sim \mathbf{a} = \sim \mathbf{a}.$$

What can we infer about

$$\underbrace{\sim \cdots \sim}_{n \text{ times}} \mathbf{a}? \qquad (n \in Z^{+})$$

How can we relate this to the repeated complement of a set? ■

11.1.2 Conjunction and Intersection; Alternation and Union

11.1.2

Main Text
* * *

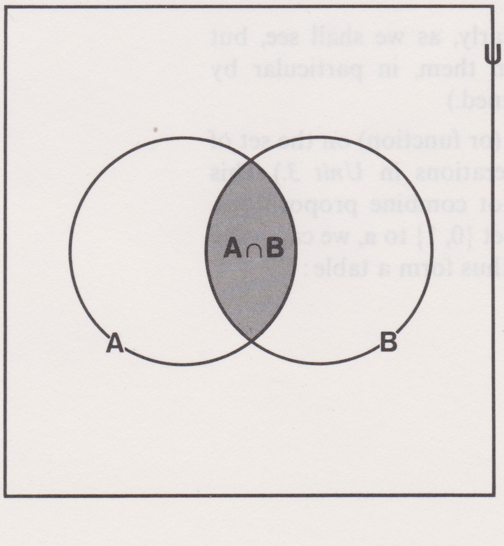
In *Unit 6, Inequalities* we defined the intersection of two sets, A and B, written

$$A \cap B,$$

as the set of elements of U which belong both to A and to B. That is:

$$A \cap B = \{x \in U : x \in A \text{ and } x \in B\}.$$

The Venn diagram depicting $A \cap B$ is:



For each element x belonging to $A \cap B$ we have:

$x \in A$ and $x \in B$.

This is a compound proposition formed by combining the two simple propositions:

$x \in A$,

$x \in B$

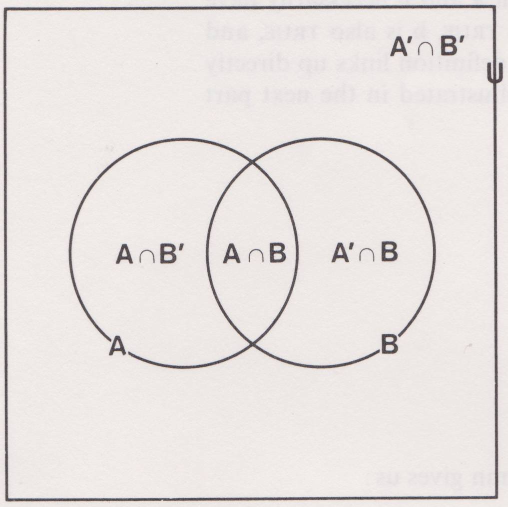
by the connective and.

Let us represent these two simple propositions by **a** and **b** respectively. The logic symbol for and is $\wedge$, so our original proposition is represented by

$\mathbf{a} \wedge \mathbf{b}$

(read as **a** and **b**).

Looking at the general Venn diagram for two sets A and B we see that it is divided into four non-overlapping regions:



For any element x in the region designated by $A' \cap B'$, we have

$x \notin A$ and $x \notin B$.

For any element x in the region designated by $A \cap B'$, we have

$x \in A$ and $x \notin B$.

For any element x in the region designated by $A' \cap B$, we have

$x \notin A$ and $x \in B$.

For any element x in the region designated by $A \cap B$, we have

$x \in A$ and $x \in B$.

Thus the four regions correspond to all the possible combinations of the truth values 0, 1 which we can allocate to the two simple propositions

$x \in A$

and

$x \in B$.

Let us now see how the truth values of the compound proposition

$x \in A$ and $x \in B$,

depend upon the truth values of the two simple propositions from which it is formed. In other words, let us make use of our understanding of the (*continued on page 9*)

Solution 11.1.1.1 **Solution 11.1.1.1**

(i) Suitable negations are:

(a) The man does not have blue eyes.

(b) $x \nleq y$, meaning x is not less than or equal to y, or x is neither less than nor equal to y, or x is greater than y.

(c) The temperature is not below 10 °C.

(d) It is foggy.

Notice that a proposition together with its negation must *exhaust all the possibilities.* Thus, to give

"the man has brown eyes"

as the *negation* of "the man has blue eyes" is incorrect. Similarly an incorrect answer for (c) would be

"the temperature is above 10°C",

since a temperature of exactly 10 °C is not mentioned.

(ii) Both "**a** equals **b**" and "**a** is the same as **b**" beg the question, since we have no measure of equality or sameness. In terms of truth values it is easier to be precise: we write $\mathbf{a} = \mathbf{b}$ if **a** and **b** necessarily have the same truth values; i.e. whenever **a** is TRUE, **b** is also TRUE, and whenever **a** is FALSE, **b** is also FALSE. This definition links up directly with our use of truth tables and this is illustrated in the next part of this exercise.

(iii) Referring to our truth table for negation:

$\mathbf{a}$	$\sim\mathbf{a}$
0	1
1	0

and writing $\sim\mathbf{a}$ for **a** in the left-hand column gives us:

$\sim\mathbf{a}$	$\sim\ \sim\mathbf{a}$
0	1
1	0

From these tables we see that when **a** is FALSE, $\sim\mathbf{a}$ is TRUE and $\sim\ \sim\mathbf{a}$ is FALSE. Also, when **a** is TRUE, $\sim\mathbf{a}$ is FALSE and $\sim\ \sim\mathbf{a}$ is TRUE.

Thus, we see that $\sim\ \sim\mathbf{a}$ *necessarily has the same truth values* as **a**.

Similarly, we obtain the truth table:

$\mathbf{a}$	$\sim\mathbf{a}$	$\sim\ \sim\mathbf{a}$	$\sim\ \sim\ \sim\mathbf{a}$
0	1	0	1
1	0	1	0

Since $\sim\mathbf{a}$ and $\sim\ \sim\ \sim\mathbf{a}$ have identical truth values, we write

$$\sim\sim\sim\mathbf{a} = \sim\mathbf{a}.$$

We can thus infer that: **Equation (1)**

$$\underbrace{\sim\cdots\sim}_{n\text{ times}}\mathbf{a} = \mathbf{a} \text{ when } n \text{ is even}$$
$$= \sim\mathbf{a} \text{ when } n \text{ is odd.}$$

Similarly, for the set A and its complement A', we have A complemented n times equal to A when n is even and to A' when n is odd. ■

(*continued from page 7*)

concept of the intersection of two sets to develop a truth table for the logical connective $\wedge$. We begin by filling in the left-hand side of our truth table with the four possible truth values of **a** and **b**, **a** denoting "$x \in A$" and **b** denoting "$x \in B$":

a	**b**	**a** $\wedge$ **b**
0	0	
0	1	
1	0	
1	1	

We interpret this table by reading along each row a set of corresponding truth values of **a**, **b** and **a** $\wedge$ **b** respectively. Thus, the first row corresponds to **a** being FALSE and **b** being FALSE; the entry for **a** $\wedge$ **b** has still to be made. We know that the compound proposition

$x \in A$ and $x \in B$

is TRUE only when each of the constituent simple propositions is TRUE, i.e. when the truth values of **a** and **b** are such that we map both **a** to 1 and **b** to 1. We can thus complete the truth table by filling in the right-hand column to give:

a	**b**	**a** $\wedge$ **b**
0	0	0
0	1	0
1	0	0
1	1	1

Now, we know further that set intersection is commutative (see *Unit 6*), i.e. $A \cap B = B \cap A$ for all sets A and B, and, as we would expect, the connective $\wedge$ is a commutative binary operation on the set of all propositions, P. This is also apparent from our truth table, since interchanging the two columns on the left of the table will not alter the column of truth values on the right. Similarly, as set intersection is associative (see *Unit 6*), i.e. $A \cap (B \cap C) = (A \cap B) \cap C$ for all sets A, B and C, we would expect to find that $\wedge$ is also an associative binary operation. This can be proved, independently of any consideration of sets, by an extended truth table

which includes *all* the eight possible combinations of truth values for **a**, **b** and **c**, where **c** represents "$x \in C$":

a	**b**	**c**	**a** $\wedge$ **b**	**b** $\wedge$ **c**	(**a** $\wedge$ **b**) $\wedge$ **c**	**a** $\wedge$ (**b** $\wedge$ **c**)
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	0	0
0	1	1	0	1	0	0
1	0	0	0	0	0	0
1	0	1	0	0	0	0
1	1	0	1	0	0	0
1	1	1	1	1	1	1

The eight lines of the first three columns of this table show these eight possible combinations of truth values. The rest of the table is built up entirely by reference to the basic table:

a	**b**	**a** $\wedge$ **b**
0	0	0
0	1	0
1	0	0
1	1	1

Thus, the fourth column in the extended truth table corresponds to the basic table above being applied to the propositions **a** and **b**. The fifth column corresponds to applying it to the two propositions **b** and **c**. The sixth column corresponds to applying the basic table to the two propositions **a** $\wedge$ **b** and **c**, and the seventh column corresponds to applying the basic table similarly to **a** and **b** $\wedge$ **c**. The last two columns are identical; that is, whenever (**a** $\wedge$ **b**) $\wedge$ **c** is TRUE, **a** $\wedge$ (**b** $\wedge$ **c**) is TRUE, and whenever (**a** $\wedge$ **b**) $\wedge$ **c** is FALSE, **a** $\wedge$ (**b** $\wedge$ **c**) is FALSE, and vice versa. So we write

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c}$$

The connective $\wedge$ is a binary operation defined on the set of all propositions, P, which is both commutative and associative on P. For $\mathbf{a}, \mathbf{b} \in P$, the proposition **a** $\wedge$ **b** is called the conjunction of the propositions **a** and **b**.

Definition 1
* * *

Exercise 1

Exercise 1
(2 minutes)

Consider the following commutative diagram:

$$\begin{tikzcd}
\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots\} \arrow[r, "\wedge"] \arrow[d, "T"'] & \{\mathbf{a} \wedge \mathbf{b}, \ldots\} \arrow[d, "T"] \\
\{0, 1\} \arrow[r, "\square"] & \{0, 1\}
\end{tikzcd}$$

Suggest an appropriate binary operation $\square$ on the set $\{0, 1\}$. (Note that $\square$ is defined by the known properties of $\wedge$ and the fact that we have drawn this commutative diagram. For instance, if **a** and **b** are both false, we have:

$$\begin{tikzcd}
(\mathbf{a}, \mathbf{b}) \arrow[r, "\wedge"] \arrow[d, "T"'] & \mathbf{a} \wedge \mathbf{b} \arrow[d, "T"] \\
(0, 0) \arrow[r, "\square"] & 0
\end{tikzcd}$$

So we know that $0 \square 0 = 0$.) ■

Looking back once more to *Unit 6*, we defined the union of two sets, A and B, written

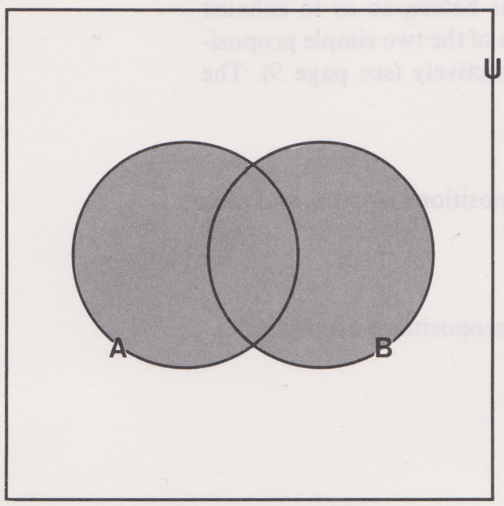
Main Text
* * *

$A \cup B,$

as the set of elements which belong either to A or to B or to both. That is:

$A \cup B = \{x \in U : x \in A \text{ or } x \in B\}$

The Venn diagram depicting $A \cup B$ is:



For each element x belonging to $A \cup B$ we have

$x \in A \quad \text{or} \quad x \in B.$

Notice that here we are using the connective "or" to include the "or both" case. This connective is given the logic symbol $\vee$, so we can represent our compound proposition by

$\mathbf{a} \vee \mathbf{b},$

where **a** represents "$x \in A$" and **b** represents "$x \in B$".

When using the informal language of everyday English, we use the connective "or" sometimes to include the "or both" case and sometimes to exclude it, and we rely on the general context to convey our meaning. For example, if you are in a restaurant and are offered chicken *or* beef as the main-course dish of a set-price lunch, the proprietor means you to understand that you can choose *either* chicken *or* beef *but not both.* On the other hand, if the proprietor offered you salt *or* pepper with your savoury course, he would hardly be disconcerted if you decided to take both. In logic, it is necessary to distinguish between these two different senses in which the word "or" is used. By definition, the connective $\vee$ refers to the "or both" sense; later in the unit we shall consider the connective "or ... but not both", which we shall denote by a different symbol.

(*continued on page 12*)

Solution 1 **Solution 1**

An appropriate binary operation on $\{0, 1\}$ is multiplication.

This gives

$$0 \times 0 = 0$$

$$0 \times 1 = 0$$

$$1 \times 0 = 0$$

$$1 \times 1 = 1$$

We thus have a *morphism*

$$T:(P, \wedge) \longmapsto (\{0, 1\}, \times)$$

and since the mapping T is many-one for given truth values of the elements of P, this morphism is a *homomorphism*. ■

(*continued from page 11*)

We now look again at the four basic non-overlapping areas of the Venn diagram, and use our understanding of the concept of the union of two sets to obtain the truth table for the logical connective $\vee$. We begin by constructing the left-hand side of the table, as before, so as to exhaust all the possible combinations of the truth values of the two simple propositions, **a**, **b** representing "$x \in A$", "$x \in B$" respectively (see page 9). The proposition

$$x \in A \quad \text{or} \quad x \in B$$

is TRUE if either of the constituent simple propositions is TRUE, and also if both are TRUE. Alternatively, we can say that

$$x \in A \quad \text{or} \quad x \in B$$

is FALSE only if both of the constituent simple propositions are FALSE.

We thus obtain:

a	**b**	**a** $\vee$ **b**
0	0	0
0	1	1
1	0	1
1	1	1

Since we know from *Unit 6* that set union is commutative and associative, we find also that the connective $\vee$ is a commutative and associative binary operation on the set of all propositions, P.

For **a**, **b** $\in P$, **a** $\vee$ **b** is called the alternation (or disjunction) of the propositions **a** and **b**. **Definition 2** ***

Exercise 2 **Exercise 2**

(i) Prove by means of a truth table that alternation is an associative operation on P. (See page 10 where we proved the associativity of conjunction.) (3 minutes)

(ii) Given a commutative diagram similar to that of Exercise 1, but with $\vee$ replacing $\wedge$, describe an appropriate binary operation $\square$ on the set of integers $\{0, 1\}$. (3 minutes) ■

Main Text
* * *

We have seen that $\sim$ is a unary operation on the set of propositions P, and that $\wedge$, $\vee$ are binary operations on P. Regarding $\sim$ as a function with domain and codomain P, and again recalling the study of morphisms (*Unit 3*), we are led to ask what happens to the two binary operations of conjunction and alternation if we use $\sim$ to map all the elements of P to their negations.

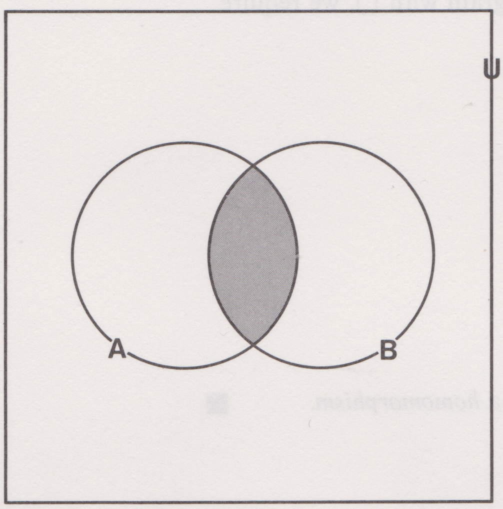
Let us start by drawing a commutative diagram as far as we can, beginning with two propositions **a**, **b** in the top left-hand corner. We obtain for the binary operation $\wedge$:

$$\begin{tikzcd} (\mathbf{a}, \mathbf{b}) \arrow[r, "\wedge"] \arrow[d, "\sim"'] & \mathbf{a} \wedge \mathbf{b} \arrow[d, "\sim"] \\ (\sim\mathbf{a}, \sim\mathbf{b}) & \sim(\mathbf{a} \wedge \mathbf{b}) \end{tikzcd}$$

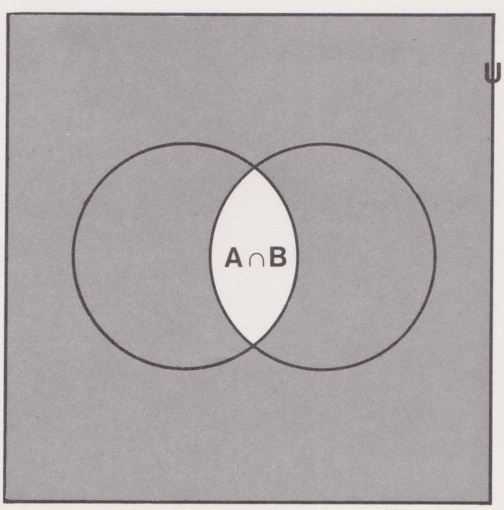
We are left with the problem of how to complete the rectangle, that is: "Is there a binary operation $\square$ (and, if so, can we recognize it among those we know?) for which $\sim\mathbf{a} \square \sim\mathbf{b} = \sim(\mathbf{a} \wedge \mathbf{b})$?" We can turn once more to our experience of sets and Venn diagrams to help us. We let **a**, **b** represent the propositions

$$x \in A, \; x \in B$$

as before. Since $\cap$ corresponds to $\wedge$, we consider the Venn diagram for $A \cap B$, the intersection of the sets A and B:



Negation corresponds to complement, and for the complement of $A \cap B$ we have:



(*continued on page 15*)

Solution 2

(i)

a	b	c	a ∨ b	b ∨ c	(a ∨ b) ∨ c	a ∨ (b ∨ c)
0	0	0	0	0	0	0
0	0	1	0	1	1	1
0	1	0	1	1	1	1
0	1	1	1	1	1	1
1	0	0	1	0	1	1
1	0	1	1	1	1	1
1	1	0	1	1	1	1
1	1	1	1	1	1	1

The last two right-hand columns are identical, hence

$$(\mathbf{a} \vee \mathbf{b}) \vee \mathbf{c} = \mathbf{a} \vee (\mathbf{b} \vee \mathbf{c})$$

(ii) In order to complete the commutative diagram with □, we require

$$0 \,\square\, 0 = 0$$
$$0 \,\square\, 1 = 1$$
$$1 \,\square\, 0 = 1$$
$$1 \,\square\, 1 = 1$$

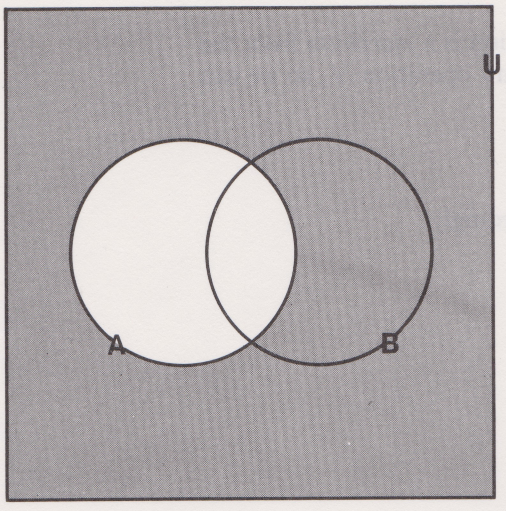
We thus have a *morphism*

$$T:(P, \vee) \longmapsto (\{0, 1\}, \square)$$

and since T is many-one, this morphism is a *homomorphism*. ■

(continued from page 13)

In order to solve our problem of finding out what $\sim(\mathbf{a} \wedge \mathbf{b})$ is, we need to express this shaded area, which represents $(A \cap B)'$, in terms of A' and B'.



A' shown shaded

We have drawn these two diagrams* so that they can be superimposed on each other, and you should be able to see that

$$(A \cap B)' = A' \cup B'.$$

That is, the complement of the intersection of A and B is the union of the complements of A and B. We are thus led to infer that

$$\sim(\mathbf{a} \wedge \mathbf{b}) = \sim\mathbf{a} \vee \sim\mathbf{b}$$

and we see that □ is to be interpreted as $\vee$.

We can readily obtain a formal proof of this result by using the truth tables for negation, conjunction and alternation as follows:

$\mathbf{a}$	$\mathbf{b}$	$\sim\mathbf{a}$	$\sim\mathbf{b}$	$\mathbf{a} \wedge \mathbf{b}$	$\sim\mathbf{a} \vee \sim\mathbf{b}$	$\sim(\mathbf{a} \wedge \mathbf{b})$
0	0	1	1	0	1	1
0	1	1	0	0	1	1
1	0	0	1	0	1	1
1	1	0	0	1	0	0

* The second diagram is Overlay 1, which you will find in the pocket at the back of this text.

As the last two columns are identical, we can write

$$\sim\mathbf{a} \vee \sim\mathbf{b} = \sim(\mathbf{a} \wedge \mathbf{b})$$

Thus, the unary operation (function) of negation is a *morphism* from the set P with binary operation $\wedge$ to P with binary operation $\vee$, so we can write

$$\sim : (P, \wedge) \longmapsto (P, \vee),$$

and complete our commutative diagram accordingly:

$$\begin{tikzcd} (\mathbf{a}, \mathbf{b}) \arrow[r, "\wedge"] \arrow[d, "\sim"'] & \mathbf{a} \wedge \mathbf{b} \arrow[d, "\sim"] \\ (\sim\mathbf{a}, \sim\mathbf{b}) \arrow[r, "\vee"] & \sim(\mathbf{a} \wedge \mathbf{b}) = \sim\mathbf{a} \vee \sim\mathbf{b} \end{tikzcd}$$

Exercise 3

Exercise 3
(3 minutes)

Prove by using truth tables that

$$\sim : (P, \vee) \longmapsto (P, \wedge)$$

is a morphism, and draw the corresponding commutative diagram for two elements $\mathbf{a}, \mathbf{b} \in P$. ■

Main Text
* * *

There is one investigation concerning the two binary operations $\wedge$, $\vee$ which we have not yet considered, and that is their respective distributivity over each other. This question always arises when we have two binary operations on a set. (See section 3.1.4 of *Unit 3*, page 14, for the definition of distributivity.) That is to say, we want to determine whether or not

$$\mathbf{a} \wedge (\mathbf{b} \vee \mathbf{c}) = (\mathbf{a} \wedge \mathbf{b}) \vee (\mathbf{a} \wedge \mathbf{c})$$

and whether or not

$$\mathbf{a} \vee (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \vee \mathbf{b}) \wedge (\mathbf{a} \vee \mathbf{c}).$$

The obvious way to see whether either of these two equations holds is to look at the corresponding truth tables.

Exercise 4

Exercise 4

(i) Prove by means of a truth table that (2 minutes)
  (a) conjunction is distributive over alternation
  and
  (b) alternation is distributive over conjunction.

(ii) Use the distributive properties of $\wedge$ and $\vee$ to reduce the following compound proposition into a simpler form in which each of the propositions $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$ occurs once only: (2 minutes)

$$[(\mathbf{a} \wedge \mathbf{b}) \vee (\mathbf{a} \wedge \mathbf{c}) \vee \mathbf{d}] \wedge [(\mathbf{a} \wedge \mathbf{b}) \vee (\mathbf{a} \wedge \mathbf{c}) \vee \mathbf{e}]$$

(iii) From the fact that $\vee$ corresponds to $\cup$ and $\wedge$ corresponds to $\cap$, we would expect that for sets: (3 minutes)
  (a) intersection is distributive over union
  and
  (b) union is distributive over intersection.
  Use Venn diagrams to verify these results. ■

11.1.3 Implication and Inclusion

11.1.3

Main Text

* * *

We are now going to consider another logical connective where, as with the word "or", ordinary English usage may not be sufficiently clear for the purpose of our study. For example, consider compound propositions such as:

"If I get a rise in salary, then I shall buy a new car";

"If 8 is divisible by 4, then 16 is divisible by 4";

"If it is sunny, then it is also warm".

We again start with an appropriate example from set theory, namely the compound proposition

"if $x \in A$, then $x \in B$".

Notation 1

* * *

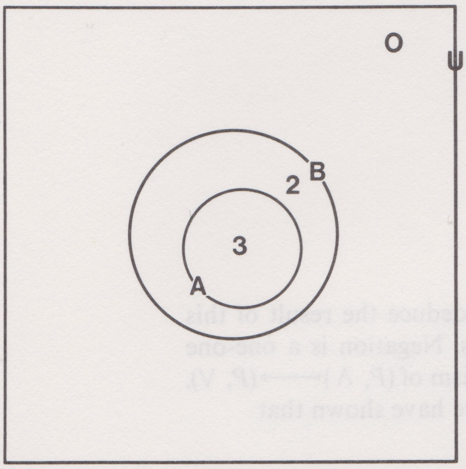
The logic symbol for "if . . . then . . ." is $\Rightarrow$ and so, defining simple propositions **a**, **b** as previously, our compound proposition becomes

$$\mathbf{a} \Rightarrow \mathbf{b}$$

which we usually read **"a implies b"**. If the compound proposition in term of sets is to be true for *every* $x \in A$, then A must be a subset of B, i.e.

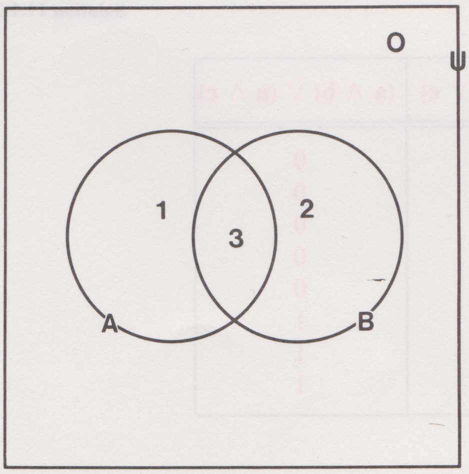
$$A \subseteq B.$$

The Venn diagram for this situation is:



(We have numbered the basic regions of the Venn diagram, 0, 2, 3 for easy reference and comparison with the diagram which follows below.)

However, let us suppose that $A \nsubseteq B$, but that the compound proposition for sets A and B is TRUE for certain $x \in A$ and FALSE for certain other $x \in A$, and let us look also at the general Venn diagram for two sets to see if this will help us to obtain a truth table for $\Rightarrow$.



(continued on page 20)

Solution 11.1.2.3

Solution 11.1.2.3

a	b	~a	~b	a ∨ b	~a ∧ ~b	~(a ∨ b)
0	0	1	1	0	1	1
0	1	1	0	1	0	0
1	0	0	1	1	0	0
1	1	0	0	1	0	0

As the last two columns are identical, we can write

$$\sim\mathbf{a} \wedge \sim\mathbf{b} = \sim(\mathbf{a} \vee \mathbf{b})$$

Thus, the operation of negation is a *morphism* from the set P with binary operation $\vee$ to P with binary operation $\wedge$, so we can write

$$\sim : (P, \vee) \longmapsto (P, \wedge),$$

and we can draw the commutative diagram:

$$\begin{tikzcd}
(\mathbf{a}, \mathbf{b}) \arrow[r, "\vee"] \arrow[d, "\sim"'] & \mathbf{a} \vee \mathbf{b} \arrow[d, "\sim"] \\
(\sim\mathbf{a}, \sim\mathbf{b}) \arrow[r, "\wedge"] & \begin{array}{c}\sim(\mathbf{a} \vee \mathbf{b}) \\ = \sim\mathbf{a} \wedge \sim\mathbf{b}\end{array}
\end{tikzcd}$$

(You may be interested to note that we could deduce the result of this exercise without going through all these details. Negation is a one-one function of P to itself and therefore an isomorphism of $(P, \wedge) \longmapsto (P, \vee)$. The inverse function of negation is itself, since we have shown that

$$\sim \sim\mathbf{a} = \mathbf{a}$$

for all $\mathbf{a} \in P$, and, since the inverse of an isomorphism is an isomorphism (see *Unit 3*, page 28), it follows that

$$\sim : (P, \vee) \longmapsto (P, \wedge)$$

is an isomorphism.) ■

Solution 11.1.2.4

Solution 11.1.2.4

(i) (a)

a	b	c	b ∨ c	a ∧ b	a ∧ c	a ∧ (b ∨ c)	(a ∧ b) ∨ (a ∧ c)
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	0	1	1	1
1	1	0	1	1	0	1	1
1	1	1	1	1	1	1	1

Hence $\mathbf{a} \wedge (\mathbf{b} \vee \mathbf{c}) = (\mathbf{a} \wedge \mathbf{b}) \vee (\mathbf{a} \wedge \mathbf{c})$

(b)

a	b	c	b ∧ c	a ∨ b	a ∨ c	a ∨ (b ∧ c)	(a ∨ b) ∧ (a ∨ c)
0	0	0	0	0	0	0	0
0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

Hence $\mathbf{a} \vee (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \vee \mathbf{b}) \wedge (\mathbf{a} \vee \mathbf{c})$

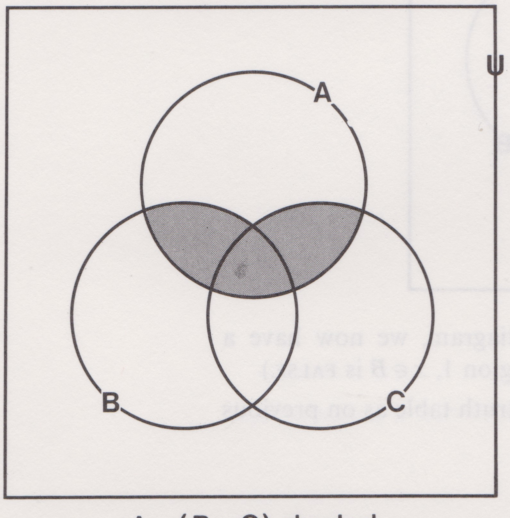
Since $\wedge$ and $\vee$ are commutative, it follows that each of $\wedge$ and $\vee$ is distributive over the other

(ii) $[(\mathbf{a} \wedge \mathbf{b}) \vee (\mathbf{a} \wedge \mathbf{c}) \vee \mathbf{d}] \wedge [(\mathbf{a} \wedge \mathbf{b}) \vee (\mathbf{a} \wedge \mathbf{c}) \vee \mathbf{e}]$

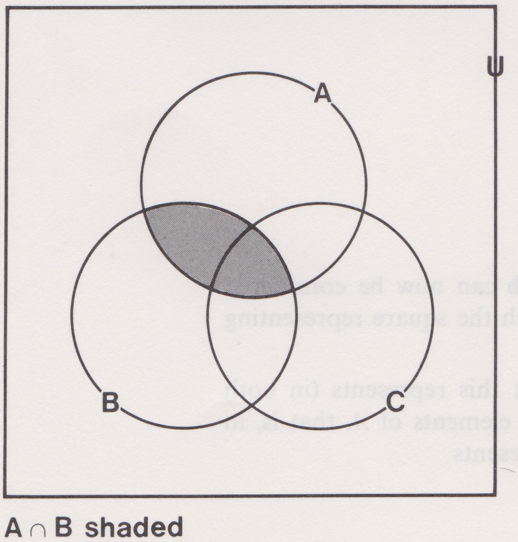
$= ([\mathbf{a} \wedge (\mathbf{b} \vee \mathbf{c})] \vee \mathbf{d}) \wedge ([\mathbf{a} \wedge (\mathbf{b} \vee \mathbf{c})] \vee \mathbf{e})$

$= [\mathbf{a} \wedge (\mathbf{b} \vee \mathbf{c})] \vee (\mathbf{d} \wedge \mathbf{e})$

(iii) The following diagrams* can be used to verify that intersection is distributive over union:



A ∩ (B ∪ C) shaded

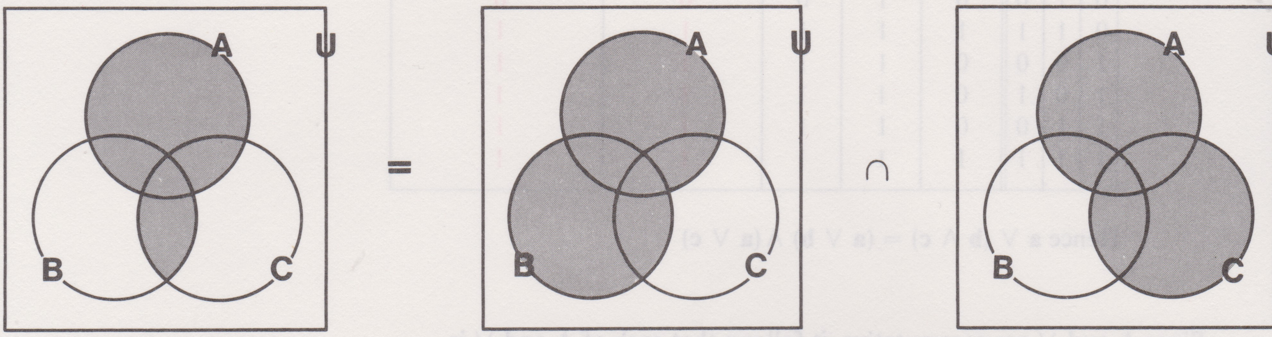


A ∩ B shaded

* The third diagram is Overlay 2, which you will find in the pocket at the back of this text.

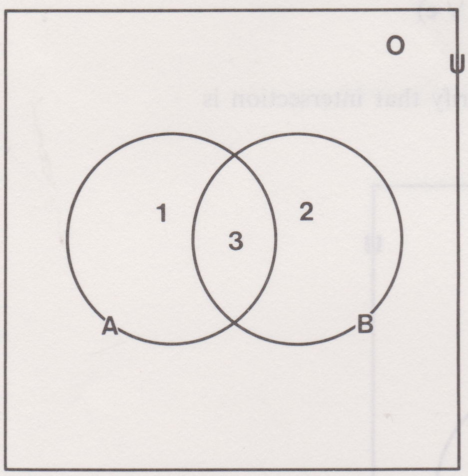
The union of the shaded regions in the last two diagrams is the same as the shaded region in the first diagram.

Similar diagrams can be used to verify the distribution of union over intersection:



$A \cup (B \cap C)$ $\qquad$ $A \cup B$ $\qquad$ ■ $\qquad$ $A \cup C$

(*continued from page 17*)



(Notice that, in contrast with the previous diagram, we now have a region 1 in addition to regions 0, 2, 3. In the region 1, $x \in B$ is FALSE.)

We begin by filling in the left-hand side of our truth table as on previous occasions:

a	**b**	**a ⇒ b**
0	0	
0	1	
1	0	
1	1	

The four combinations of truth values of **a**, **b** can now be considered in the light of the four basic regions into which the square representing the universal set is divided.

If we consider first the region 3, we see that this represents (in both diagrams) those elements of B which are also elements of A, that is, in terms of our mapping T to truth values, it represents

$T: \mathbf{a} \longmapsto 1,$

$T: \mathbf{b} \longmapsto 1,$

and for all elements of this region $\mathbf{a} \Rightarrow \mathbf{b}$ is TRUE, i.e. it is true that if $x \in A$ then $x \in B$.

We can thus complete the bottom line of our truth table by entering 1 for TRUE in the right-hand column.

Next consider the region 1, which we find only in the second diagram. This corresponds to

$$T:\mathbf{a} \longmapsto 1,$$

$$T:\mathbf{b} \longmapsto 0.$$

All elements of region 1 are members of A but not of B, and hence our compound proposition $\mathbf{a} \Rightarrow \mathbf{b}$ is FALSE for all elements of A in region 1. We can thus complete the third line of our truth table by entering 0 in the right-hand column.

The two cases which we have just considered are in accord with our normal interpretation of "if ... then ..." in everyday conversation. Thus, considering the proposition:

"If I get a rise in salary, then I shall buy a new car",

we accept this as verified (TRUE) if the rise in salary materialises (TRUE) and the new car is bought (TRUE), and as disproved (FALSE) if the rise materialises (TRUE) but the new car is not bought (FALSE).

We are now left with the first and second lines of our truth table, which represent the two possible cases where **a** is FALSE. The truth value of the proposition $\mathbf{a} \Rightarrow \mathbf{b}$ is now not so immediately obvious. However, reference back to our two Venn diagrams does help us, and we are able to see that the existence of elements in regions 0 and 2 does not preclude all the elements of A being also elements of B. Elements of region 0 correspond to

$$T:\mathbf{a} \longmapsto 0$$

$$T:\mathbf{b} \longmapsto 0$$

and elements of region 2 correspond to

$$T:\mathbf{a} \longmapsto 0$$

$$T:\mathbf{b} \longmapsto 1.$$

Since in these cases $\mathbf{a} \Rightarrow \mathbf{b}$ still holds, we complete our truth table by entering 1 for TRUE in the remaining places in the right-hand column. Thus the truth table for $\Rightarrow$ is

a	b	$\mathbf{a} \Rightarrow \mathbf{b}$
0	0	1
0	1	1
1	0	0
1	1	1

(This may all seem rather arbitrary to you. In fact we could have spent a long time arguing out that $\Rightarrow$ is the connective having this particular truth table. But we chose to offer some explanation for the truth table adopted for $\Rightarrow$ by reference to sets. We shall discuss the construction of the truth table for this connective again in our radio programme.)

The connective $\Rightarrow$ is called the connective of implication. It is because we tend to read the idea of cause and effect into statements such as

Definition 1
* * *

"If I receive an increase in salary, then I shall buy a new car",

that the truth table for implication may seem a little strange at first. We must remember, however, that $\Rightarrow$ should be regarded simply as a logical connective. That is to say, we are not to interpret $\mathbf{a} \Rightarrow \mathbf{b}$ as

"**a** causes **b**"

or as

"**b** results from **a**"

but merely as the compound proposition resulting from the operation of the connective $\Rightarrow$ on the *ordered pair* $\mathbf{a}, \mathbf{b} \in P$.

Of course, if we know from external considerations that there really is a cause and effect relationship, we are entitled to deduce that $\mathbf{a} \Rightarrow \mathbf{b}$ is TRUE, i.e. that the truth values of **a** and **b** must be such that $\mathbf{a} \Rightarrow \mathbf{b}$ is TRUE. On the other hand, we cannot deduce that a cause and effect relationship exists merely from the acceptance that $\mathbf{a} \Rightarrow \mathbf{b}$ is TRUE. For example, looking at the two propositions:

"If I receive a rise in salary, then I shall buy a new car",

and

"If Shakespeare wrote Hamlet, then Stilton is a cheese",

we note that they could have exactly the same form of logical representation, $\mathbf{a} \Rightarrow \mathbf{b}$. In both cases, we could have, for instance $\mathbf{a}, \mathbf{b}, \mathbf{a} \Rightarrow \mathbf{b}$ all having the truth value 1, but we would hardly infer any causal relationship in the latter case. To make a positive assertion of a cause and effect relationship is to assert that, if some proposition **a** is true, then of necessity *as a result* proposition **b** is also true. This is much stronger than the situation which we have with $\mathbf{a} \Rightarrow \mathbf{b}$, but it is not inconsistent with it.

Exercise 1

Exercise 1

If you are satisfied that you can manipulate truth tables easily, there is no need to attempt both parts of questions (i) and (iii). You should, however, take note of the solutions to these questions.

(i) Is the binary operation $\Rightarrow$ on the set of propositions P (4 minutes)
 (a) commutative?
 (b) associative?

(ii) Given that (3 minutes)

$$\sim :(P, \Rightarrow) \longmapsto (P, \square)$$

is a *morphism*, find the truth table for the logical connective represented by $\square$.

(iii) is $\Rightarrow$ left-distributive over (2 minutes)
 (a) $\wedge$?
 (b) $\vee$?

(iv) Show that (2 minutes)

$$\mathbf{a} \Rightarrow \mathbf{b} = \sim \mathbf{a} \vee \mathbf{b}$$

and give an example with propositions stated in words. (N.B. The negation refers to the proposition **a** only. It is conventional to omit the brackets, as we have done, in an expression such as $(\sim \mathbf{a}) \vee \mathbf{b}$.)

■

11.1.4 Equivalence and Equality

11.1.4

Main Text

★★

You are no doubt familiar with the two terms "equivalence" and "equality". In different circumstances they appear with meanings sometimes virtually identical, sometimes meaningfully different. In the propositional calculus, there is a connective of equivalence, and we now discuss this, taking as a starting point the equality of sets.

In *Unit 1* we defined two sets, A, B, to be *equal* if they each comprise exactly the same elements. We can express this equality of sets in terms of our propositions **a**, **b**, previously defined to be "$x \in A$", "$x \in B$" respectively. If $A = B$ we have:

if $x \in A$, then $x \in B$

and

if $x \in B$, then $x \in A$;

that is,

$\mathbf{a} \Rightarrow \mathbf{b}$

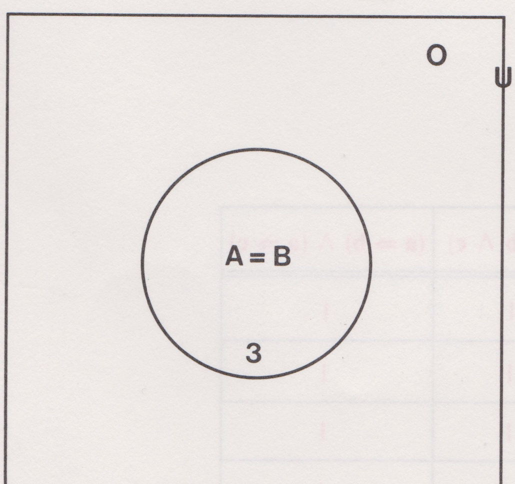
and

$\mathbf{b} \Rightarrow \mathbf{a}$

must both be TRUE. In seeking for a logical connective corresponding to this situation, we again start with our embryo truth table, in which we adopt the symbol $\Leftrightarrow$ to represent the connective we are looking for:

a	b	$\mathbf{a} \Leftrightarrow \mathbf{b}$
0	0	
0	1	
1	0	
1	1	

The Venn diagram corresponding to the equality of sets A and B is:



and we note that regions 1 and 2 from the general Venn diagram are not present. The two regions 0, 3 correspond to:

$T: \mathbf{a} \longmapsto 0$

$T: \mathbf{b} \longmapsto 0$

(*continued on page 25*)

Solution 11.1.3.1

(i) (a) NO

For instance, the table for "$\mathbf{b} \Rightarrow \mathbf{a}$" is

a	**b**	**b ⇒ a**
0	0	1
0	1	0
1	0	1
1	1	1

and the third column here is not the same as that in the table for "$\mathbf{a} \Rightarrow \mathbf{b}$". Consider also "If I buy a new car, then I shall get a rise in salary".

(b) NO

As a counter-example, consider the propositions $\mathbf{a} \Rightarrow (\mathbf{b} \Rightarrow \mathbf{c})$, $(\mathbf{a} \Rightarrow \mathbf{b}) \Rightarrow \mathbf{c}$ for the case in which **a**, **b**, **c** are all FALSE.

Then $\mathbf{b} \Rightarrow \mathbf{c}$ and $\mathbf{a} \Rightarrow \mathbf{b}$ are both TRUE, and so $\mathbf{a} \Rightarrow (\mathbf{b} \Rightarrow \mathbf{c})$ is TRUE, but $(\mathbf{a} \Rightarrow \mathbf{b}) \Rightarrow \mathbf{c}$ is FALSE. So $\mathbf{a} \Rightarrow (\mathbf{b} \Rightarrow \mathbf{c}) \neq (\mathbf{a} \Rightarrow \mathbf{b}) \Rightarrow \mathbf{c}$.

(ii)

a	**b**	**a □ b**
0	0	0
0	1	1
1	0	0
1	1	0

We can obtain the entries in this table by using a commutative diagram; e.g. the last row is obtained from

$$\begin{tikzcd} (0,0) \arrow[r, "\Rightarrow"] \arrow[d, "\sim"'] & 1 \arrow[d, "\sim"] \\ (1,1) \arrow[r, "\square"] & 0 \end{tikzcd}$$

(iii) (a) YES

The complete truth table is:

a	**b**	**c**	**a ⇒ b**	**a ⇒ c**	**b ∧ c**	**a ⇒ (b ∧ c)**	**(a ⇒ b) ∧ (a ⇒ c)**
0	0	0	1	1	0	1	1
0	0	1	1	1	0	1	1
0	1	0	1	1	0	1	1
0	1	1	1	1	1	1	1
1	0	0	0	0	0	0	0
1	0	1	0	1	0	0	0
1	1	0	1	0	0	0	0
1	1	1	1	1	1	1	1

(b) YES

The complete truth table is:

a	b	c	$a \Rightarrow b$	$a \Rightarrow c$	$b \vee c$	$a \Rightarrow (b \vee c)$	$(a \Rightarrow b) \vee (a \Rightarrow c)$
0	0	0	1	1	0	1	1
0	0	1	1	1	1	1	1
0	1	0	1	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	0	0	0	0	0
1	0	1	0	1	1	1	1
1	1	0	1	0	1	1	1
1	1	1	1	1	1	1	1

(iv) We have:

a	b	$\sim a$	$\sim a \vee b$	$a \Rightarrow b$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

A suitable example is:

"If it is raining, then I shall get wet"

and

"It is not raining or I shall get wet" ■

(*continued from page 23*)

and

$T{:}\mathbf{a} \longmapsto 1$

$T{:}\mathbf{b} \longmapsto 1$

respectively, for which $\mathbf{a} \Leftrightarrow \mathbf{b}$ is TRUE. The two missing regions 1 and 2, correspond to

$T{:}\mathbf{a} \longmapsto 1$

$T{:}\mathbf{b} \longmapsto 0$

and

$T{:}\mathbf{a} \longmapsto 0$

$T{:}\mathbf{b} \longmapsto 1$,

respectively, and in these two last cases **a** and **b** have different truth values and so $\mathbf{a} \Leftrightarrow \mathbf{b}$ is FALSE. The truth table for $\Leftrightarrow$ is thus:

a	b	$\mathbf{a} \Leftrightarrow \mathbf{b}$
0	0	1
0	1	0
1	0	0
1	1	1

The overall proposition $\mathbf{a} \Leftrightarrow \mathbf{b}$ is TRUE whenever its constituent propositions have the same truth value, and FALSE whenever they have different truth values. The connective $\Leftrightarrow$ is read as "if and only if" or "implies and is implied by", and is the logical connective of equivalence.

Definition 1

(Notice that the symbol $\Leftrightarrow$ adopted for equivalence is derived from the symbol $\Rightarrow$ for implication, and equivalence can be thought of as a "double implication" or an "implication both ways". If $\mathbf{a} \Rightarrow \mathbf{b}$ and $\mathbf{b} \Rightarrow \mathbf{a}$, we can expect to interchange **a** and **b** in the appropriate truth table without affecting the truth of the overall compound proposition. The table adopted for $\Leftrightarrow$ satisfies just this requirement.)

Now we have used the well-known mathematical symbol "=" earlier in this unit (see, for example, page 8) to denote that two propositions necessarily have the same truth value. We have also used the concept of the equality of sets in deriving the truth table for the connective of equivalence. We are therefore naturally led to ask whether there is any difference between our interpretation of

$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}),$$

asserting the associativity of conjunction, and the interpretation which we can now give to the compound proposition:

$$[(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c}] \Leftrightarrow [\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})].$$

To appreciate that there is a difference, we recall what we said about the connective $\Rightarrow$. We made the point that $\Rightarrow$ should be regarded simply as a logical connective (page 22). This must also apply to $\Leftrightarrow$. In connecting two propositions **a**, **b** by means of $\Leftrightarrow$ we allow that **a**, **b** may have any of the possible combinations of the truth values 0, 1, and provide a truth table accordingly. In connecting two propositions by =, we are automatically excluding the two cases where the propositions differ in truth value. In other words, the assertion

$$\mathbf{a} = \mathbf{b}$$

means the same as

$$T:(\mathbf{a} \Leftrightarrow \mathbf{b}) \longmapsto 1.$$

This may perhaps seem a rather over-subtle distinction, but it is an important one; it highlights the need for a language in which to talk about propositions and the propositional calculus over and above the formal language of the propositional calculus itself. It also again highlights the difference between a *proposition*, which *may be either* TRUE *or* FALSE, and an *assertion*, which states that a given proposition *is* TRUE *or* FALSE.

Exercise 1

Exercise 1
(See page 22)

If you are satisfied with your proficiency with truth tables, there is no need to do parts (i) and (iii). You should, however, take note of the solutions to these questions.

(i) Is the binary operation $\Leftrightarrow$ on the set of propositions P (2 minutes)
  (a) commutative?
  (b) associative?

(ii) Given that $\sim :(P, \Leftrightarrow) \longmapsto (P, \square)$ is a morphism, find the truth table for the logical connective represented by $\square$. (2 minutes)

(iii) Is $\Leftrightarrow$ distributive over (2 minutes)
  (a) $\wedge$?
  (b) $\vee$?

■

11.1.5 Three Further Connectives

11.1.5

Main Text

* * *

If we start with our embryo truth table

a	b	a ? b
0	0	
0	1	
1	0	
1	1	

we can complete the right-hand column in $2^4 (= 16)$ possible ways since we can have either 0 or 1 in each place in the column, and there are four places to be filled. So far we have considered the connectives which are represented by 4 out of these 16 possible right-hand columns.

If time and space permitted, we could discuss all the sixteen theoretically possible connectives. However, we have decided to be selective, not only for the sake of brevity, but also because a number of these possible connectives are trivial in so far as they do not help us in the development of the propositional calculus. For example, the connective which would arise from the truth table:

a	b	a ? b
0	0	0
0	1	0
1	0	1
1	1	1

gives us a compound proposition which has identically the same truth values as **a**.

In this sub-section we shall complete our discussion of various connectives by considering just three more. We shall see later that two of these have special significance, and the third is one which we have already mentioned.

(*continued on page 28*)

Solution 11.1.4.1

Solution 11.1.4.1

(i) (a) YES
    (b) YES

(ii)

a	b	a □ b
0	0	0
0	1	1
1	0	1
1	1	0

(iii) (a) NO
    (b) NO

(continued from page 27)

First, consider the truth table:

a	b	a ↓ b
0	0	1
0	1	0
1	0	0
1	1	0

This is the table for the compound proposition

"neither **a** nor **b**"

which we write as

a ↓ b

Notation 1
* * *

When we compare this with the table for $\vee$ we find that the right-hand column of each is obtainable from the other by replacing 0 by 1 and 1 by 0. Whenever two columns are related in that 0 and 1 are interchanged throughout, then we have two connectives, each of which is the negation of the other. Thus the connective ↓ is the negation of $\vee$, and we can draw a triangular commutative diagram:

$$\begin{tikzcd}
(\mathbf{a}, \mathbf{b}) \arrow[r, "\vee"] \arrow[dr, "\downarrow"'] & \mathbf{a} \vee \mathbf{b} \arrow[d, "\sim"] \\
 & \sim(\mathbf{a} \vee \mathbf{b}) = \mathbf{a} \downarrow \mathbf{b}
\end{tikzcd}$$

This demonstrates diagrammatically that, for example,

"it is neither raining nor sunny"

is propositionally equivalent to

"it is not either raining or sunny".

Similarly, $\vee$ is the negation of $\downarrow$ and we can draw:

$$\begin{tikzcd}
(a, b) \arrow[r, "\downarrow"] \arrow[dr, "\vee"'] & a \downarrow b \arrow[d, "\sim"] \\
& \sim(a \downarrow b) = a \vee b
\end{tikzcd}$$

Secondly, consider the truth table:

a	b	a | b
0	0	1
0	1	1
1	0	1
1	1	0

This is the table for the compound proposition

"either not-**a** or not-**b**"

(where "or" is used in the inclusive sense "or ... or both"), which we write as

a | b.

Notation 2
* * *

Finally, consider the truth table:

a	b	a $\oplus$ b
0	0	0
0	1	1
1	0	1
1	1	0

This is the table for the compound proposition

"either **a** or **b** but not both",

which we write as

a $\oplus$ b.

Notation 3
* * *

When "or" is used in the sense of "or ... but not both", as here, it is usually called the exclusive "or" to distinguish it from the inclusive "or" above. In ordinary English usage one does not always make this distinction as it is usually clear from the context. (See page 11 for our earlier discussion.) The symbol $\oplus$ has been deliberately chosen to suggest some affinity with the + of ordinary arithmetic, and reference to the truth table will illuminate this. Each entry in the right-hand column of the table is the remainder when the sum of the two numbers on its left is divided by two. The exclusive "or" connective is thus equivalent to what we call addition

Definition 1
*

modulo two in arithmetic. The binary operation addition modulo two is denoted by $\oplus_2$; here we have:

$$0 \oplus_2 0 = 0$$

$$0 \oplus_2 1 = 1$$

$$1 \oplus_2 0 = 1$$

$$1 \oplus_2 1 = 0$$

We can draw an appropriate commutative diagram for $\mathbf{a} \longmapsto 1$, $\mathbf{b} \longmapsto 1$ (say) under T as follows:

$$\begin{tikzcd}
(\mathbf{a}, \mathbf{b}) \arrow[r, "\oplus"] \arrow[d, "T"'] & \mathbf{a} \oplus \mathbf{b} \arrow[d, "T"] \\
(1, 1) \arrow[r, "\oplus_2"'] & 0
\end{tikzcd}$$

Our mapping T is thus a homomorphism:

$$T:(P, \oplus) \longmapsto (\{0, 1\}, \oplus_2).$$

We shall see in the next sub-section that we do not need all the connectives which we have discussed despite their individual usefulness; some of them are redundant.

Discussion ★★

We end this sub-section by returning to the hypothetical problem of the traveller which we described in the introduction to this text on page x. You will recall that in order to know which way to turn at the T-junction, our traveller is allowed to ask one question only of a local inhabitant, who may only answer "Yes" or "No" and who may be a truth-teller or a liar. The problem is to determine what question our traveller should ask in order to ensure that from the answer he will know whether to turn left or to turn right in order to reach his destination.

We can tackle this by setting up the left-hand side of a truth table where **a** represents the proposition:

"the villager questioned is a truth-teller"

and **b** represents the proposition:

"the traveller should turn left to reach his destination".

The problem will be solved if a question is asked to which either a liar or a truth-teller will answer *in one particular way*, say "Yes" if turning left is correct and "No" if turning right is correct. Thus, *irrespective of the truth value of* **a**, we must get a "No" if the truth value of **b** is 0 and "Yes" if the truth value of **b** is 1. This situation is represented by:

a	b	answer
0	0	No
0	1	Yes
1	0	No
1	1	Yes

We now translate the right-hand column into the truth values for a compound proposition

a ? **b**.

But we must remember that $T:\mathbf{a} \longmapsto 0$ means that the villager giving the reply is a liar, and so, when we enter the truth values for the overall proposition in the right-hand column, we must take this into account in the first two entries. This gives us:

a	b	a ? b
0	0	
0	1	
1	0	0
1	1	1

which is the truth table for equivalence. A question which our traveller can ask is therefore:

> "Are you a truth-teller if and only if I should turn left to get to my destination X?"

This is a clumsy and rather obscure way of putting the question, though it does correspond closely to what the truth table suggests. The English can be simplified and the question rephrased as:

> "Are the two statements:
> 'I should turn left' and
> 'You are a truth-teller'
> either both TRUE or both FALSE?"

If the traveller should turn left, then

(i) a truth-teller will answer "Yes";
(ii) a liar should answer "No", but because he is a liar, he will also answer "Yes".

If the traveller should turn right, then

(i) a truth-teller will answer "No";
(ii) a liar should answer "Yes", but because he is a liar, he will also answer "No".

So, if the answer to his question is "Yes", the traveller should turn to the left, and, if the answer is "No", he should turn to the right.

Exercise 1

Exercise 1

(i) Given that $\mathbf{a}$ represents "$x \in A$" and $\mathbf{b}$ represents "$x \in B$", which regions of the general Venn diagram for the two sets correspond to (3 minutes)
   (a) $\mathbf{a} \downarrow \mathbf{b}$?
   (b) $\mathbf{a} \mid \mathbf{b}$?
   (c) $\mathbf{a} \oplus \mathbf{b}$?

(ii) Of which other connective is (4 minutes)
   (a) $\mid$
   (b) $\oplus$
   the negation? In each case, draw an appropriate triangular commutative diagram.

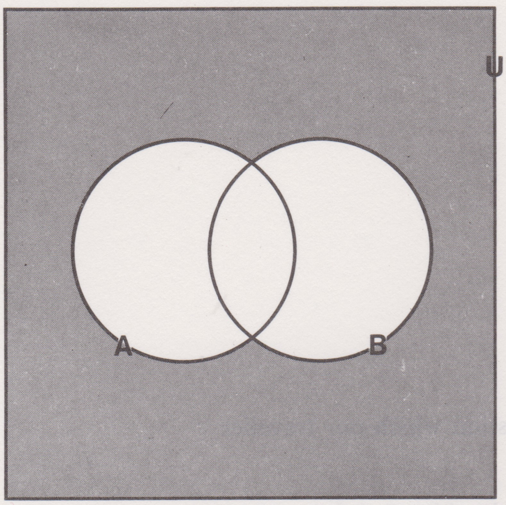
(iii) In the case of our traveller journeying to his destination X, determine whether or not the following question would produce the required result: (2 minutes)

> "If I were to ask you if I should turn left, would you say 'Yes'?" ■

Solution 1

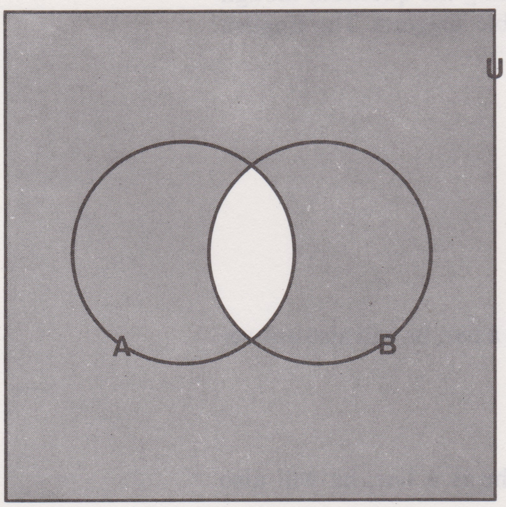
(i) The required regions are shaded

(a)



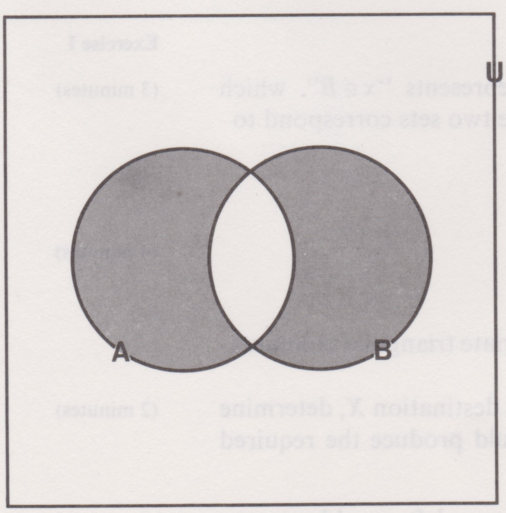
$A' \cap B'$

(b)



$A' \cup B'$

(c)



$(A' \cap B) \cup (A \cap B')$

(ii) (a)

$$\begin{tikzcd}[column sep=large, row sep=large] (\mathbf{a},\mathbf{b}) \arrow[r, "\mid"] \arrow[dr, "\wedge"'] & \mathbf{a} \mid \mathbf{b} \arrow[d, "\sim"] \\ & \sim(\mathbf{a} \mid \mathbf{b}) = \mathbf{a} \wedge \mathbf{b} \end{tikzcd}$$

$\mid$ is the negation of $\wedge$.

(or, alternatively, with $\mid$ and $\wedge$ interchanged).

(b)

$$\begin{tikzcd}[column sep=large, row sep=large] (\mathbf{a},\mathbf{b}) \arrow[r, "\oplus"] \arrow[dr, "\Leftrightarrow"'] & \mathbf{a} \oplus \mathbf{b} \arrow[d, "\sim"] \\ & \sim(\mathbf{a} \oplus \mathbf{b}) = \mathbf{a} \Leftrightarrow \mathbf{b} \end{tikzcd}$$

$\oplus$ is the negation of $\Leftrightarrow$.

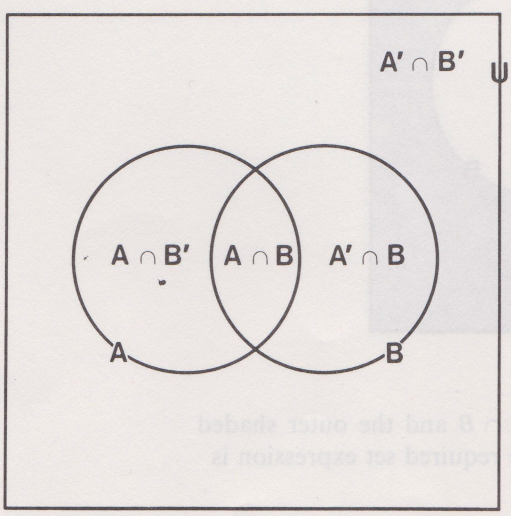
(or, alternatively, with $\oplus$ and $\Leftrightarrow$ interchanged).

(iii) The question would produce the required result. This can be seen by referring back to the truth table on page 31. A liar (i.e. $T: \mathbf{a} \longmapsto 0$) will simply reverse the truth value of **b**, but, *because he is a liar*, when answering the question as to what he would say he will deny this, and the situation will revert to that of the table on page 30. ■

11.1.6 Adequate Sets of Connectives

11.1.6
Discussion
★★

We now go on to ask a new question: Given certain logical connectives, are any of them redundant? That is, is there a subset of the set of given connectives consisting of elements in terms of which the remaining connectives can be represented? One way to approach this is via the analogy with set algebra. We have seen how the operations of set union and intersection, together with complementation, correspond to the logical connectives of alternation and conjunction, together with negation. Any of the four basic regions in the Venn diagram for two sets, shown below, can be expressed in terms of union, intersection and complementation of the sets A and B:



Further, it is possible to express any combination of some or all of these regions in terms of the same set operations on A and B, and this principle can be extended to any number of sets $A, B, C, \ldots$. Since set theory can

be built up from union, intersection and complementation, we are led to ask how far we can go towards building up a complete propositional calculus given "and", "or" and "not" only. Putting the same question in different words, we ask: "Is $\{\wedge, \vee, \sim\}$ an adequate set of connectives?" The answer to this question is "Yes", and it is possible to verify this by reference to truth tables.

Definition 1
*

Suppose, for example, that we wish to express $\mathbf{a} \Rightarrow \mathbf{b}$ in terms of some or all of $\wedge$, $\vee$, $\sim$. The truth table for the connective $\Rightarrow$ is:

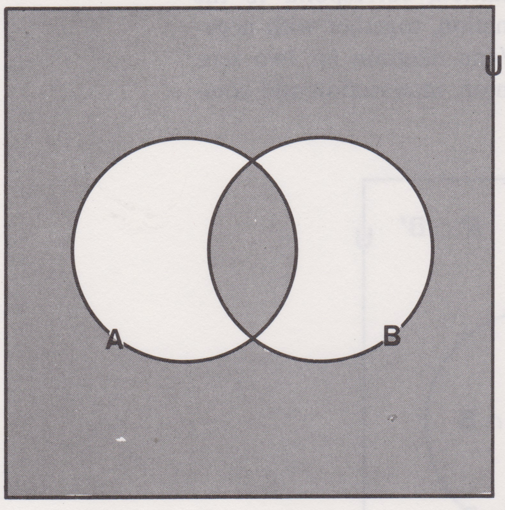
a	**b**	$\mathbf{a} \Rightarrow \mathbf{b}$
0	0	1
0	1	1
1	0	0
1	1	1

We saw in Exercise 11.1.3.1, part (iv), on page 22 that we can express $\mathbf{a} \Rightarrow \mathbf{b}$ as

$$\sim \mathbf{a} \vee \mathbf{b}.$$

(Remember that the symbol $\sim$ is tied only to the proposition immediately on its right unless brackets indicate otherwise.)

An alternative approach, which you may prefer, is to solve the corresponding problem for sets, translate the resulting expression back into logic symbols, and then verify the result by constructing the truth table. For instance, the Venn diagram corresponding to $\mathbf{a} \Leftrightarrow \mathbf{b}$ is:



The small shaded region in the middle is $A \cap B$ and the outer shaded region can be written as $(A \cup B)'$. So that the required set expression is

$$(A \cup B)' \cup (A \cap B),$$

which translated back into logical symbols becomes

$$\sim(\mathbf{a} \vee \mathbf{b}) \vee (\mathbf{a} \wedge \mathbf{b})$$

which you can verify has the same truth values as $\mathbf{a} \Leftrightarrow \mathbf{b}$.

Exercise 1

Exercise 1
(5 minutes)

Continue the verification that $\{\wedge, \vee, \sim\}$ is an adequate set of connectives by expressing $\mathbf{a} \downarrow \mathbf{b}$ and $\mathbf{a} \mid \mathbf{b}$ in terms of $\wedge$, $\vee$, $\sim$ only. ■

(We have no intention of completing the verification of all 16 possible connectives: given time, it can easily be done, but we see little educational value in asking you to perform such a tedious exercise.)

Discussion
* *

We now ask if other adequate sets of connectives exist, and, further, whether or not there is one (or more) of the connectives which is so general that all the others can be constructed from it.

If we admit the use of $\sim$, then several of the connectives can be used with it alone to form expressions equivalent to those normally formed by others. For example, we can reduce the adequate set which we have already considered, namely $\{\wedge, \vee, \sim\}$ to either $\{\wedge, \sim\}$ or $\{\vee, \sim\}$. Thus, in the first case $\mathbf{a} \vee \mathbf{b}$ becomes

$$\sim(\sim\mathbf{a} \wedge \sim\mathbf{b}),$$

and in the second case $\mathbf{a} \wedge \mathbf{b}$ becomes

$$\sim(\sim\mathbf{a} \vee \sim\mathbf{b}).$$

Definition 2
* *

Since $\{\wedge, \vee, \sim\}$ is an adequate set of connectives, it follows that $\{\wedge, \sim\}$ and $\{\vee, \sim\}$ are also adequate sets. (This, of course, implies that either $\{\cap, '\}$ or $\{\cup, '\}$ would be adequate for set algebra.) In fact, there are six connectives, each of which, together with $\sim$, form an adequate set. There are various pairs of connectives, excluding $\sim$, from which we can construct the propositional calculus, and there are just two connectives from either of which alone we can form all the others. These two are the connectives $\downarrow$ and $\mid$, that is, "neither **a** nor **b**" and "either not-**a** or not-**b**" respectively. Unfortunately these primitive connectives (as they are called) are rather clumsy when translated into everyday speech, though they are of value in the design of switching systems (which we discuss in section 11.2 and in the television programme associated with this unit). We shall again assume that $\{\wedge, \vee, \sim\}$ is an adequate set of connectives, and give the representation of these three connectives in terms of $\downarrow$:

$\mathbf{a} \wedge \mathbf{b}$ becomes $(\mathbf{a} \downarrow \mathbf{a}) \downarrow (\mathbf{b} \downarrow \mathbf{b})$

$\mathbf{a} \vee \mathbf{b}$ becomes $(\mathbf{a} \downarrow \mathbf{b}) \downarrow (\mathbf{a} \downarrow \mathbf{b})$

$\sim \mathbf{a}$ becomes $\quad \mathbf{a} \downarrow \mathbf{a}$

These results can be verified by writing out the appropriate truth tables.

Solution 1 **Solution 1**

This is no unique answer: possible answers are

$$\mathbf{a} \downarrow \mathbf{b} = \sim\mathbf{a} \wedge \sim\mathbf{b} = \sim(\mathbf{a} \vee \mathbf{b})$$

$$\mathbf{a} \mid \mathbf{b} = \sim\mathbf{a} \vee \sim\mathbf{b} = \sim(\mathbf{a} \wedge \mathbf{b}).$$ ■

11.1.7 Tautologies and Contradictions

11.1.7

Main Text

Definition 1

A compound proposition which is always true, no matter what the truth values of its constituent simple propositions are, is called a tautology. This means that the right-hand column of the extended truth table for such a compound proposition will consist of 1's.

One of the simplest of all tautologies is the proposition known as "the law of the excluded middle", which asserts

"either **a** is TRUE or not-**a** is TRUE",

which is a fundamental condition for the two-valued logic which we are considering. We can set up the appropriate truth table by reference to the truth table for $\vee$ as follows:

a	$\sim$**a**	**a** $\vee$ $\sim$**a**
0	1	1
1	0	1

Another equally simple example is provided by the compound proposition:

$$\mathbf{a} \Leftrightarrow \sim\sim\mathbf{a}.$$

The appropriate truth table is:

a	$\sim$**a**	$\sim\sim$**a**	**a** $\Leftrightarrow$ $\sim\sim$**a**
0	1	0	1
1	0	1	1

As a further example, we consider the transitivity of the logical connective of implication. (The meaning of "transitivity" here is the same as its meaning for the inequality relation $<$ which, as we saw in *Unit 6*, possesses the transitive property expressed by

$$x < y \text{ and } y < z \text{ implies } x < z \qquad (x, y, z \in R).)$$

The proposition

"implication is transitive"

which we wish to investigate as a possible tautology may be expressed as

$$[(\mathbf{a} \Rightarrow \mathbf{b}) \wedge (\mathbf{b} \Rightarrow \mathbf{c})] \Rightarrow (\mathbf{a} \Rightarrow \mathbf{c})$$

that is

"if **a** implies **b** and **b** implies **c**, then **a** implies **c**".

Exercise 1

Exercise 1
(2 minutes)

Complete the truth table below:

a	b	c	$a \Rightarrow b$	$b \Rightarrow c$	$a \Rightarrow c$	$(a \Rightarrow b) \wedge (b \Rightarrow c)$	$[(a \Rightarrow b) \wedge (b \Rightarrow c)] \Rightarrow (a \Rightarrow c)$
0	0	0	1	1	1		
0	0	1	1	1	1		
0	1	0	1	0	1		
0	1	1	1	1	1		
1	0	0	0	1	0		
1	0	1	0	1	1		
1	1	0	1	0	0		
1	1	1	1	1	1		

■

Main Text
* *

We see from the right-hand column of the solution to Exercise 1 that the proposition is true whatever the truth values of the constituent propositions **a**, **b**, **c**; it is thus a tautology and the transitivity of $\Rightarrow$ can therefore be asserted as a logical theorem, provable by the truth table above.

In earlier sub-sections of this unit we proved a number of theorems relating to the distributivity of certain connectives over others. For example, in sub-section 11.1.2 we proved that alternation is distributive over conjunction. In exactly the same way as above we can show that the compound proposition

$$a \vee (b \wedge c) \Leftrightarrow (a \vee b) \wedge (a \vee c)$$

is a tautology, thus again proving the distributivity of alternation over conjunction.

If the right-hand column of a truth table for a compound proposition consists entirely of 0's, then we have a proposition which is false whatever the truth values of its constituent simple propositions. Such a proposition is called a contradiction. One of the simplest examples is the proposition:

Definition 2
* *

$$a \wedge \sim a \qquad \text{("a and not-a")}$$

which asserts the conjunction of **a** and its negation. The truth table is:

a	$\sim a$	$a \wedge \sim a$
0	1	0
1	0	0

Other examples can be readily obtained by writing down the negation of any known theorem of logic. This has the effect simply of translating the right-hand column of 1's for the theorem into a column of 0's for its

(*continued on page 38*)

Solution 1

Solution 1

a	b	c	$a \Rightarrow b$	$b \Rightarrow c$	$a \Rightarrow c$	$(a \Rightarrow b) \wedge (b \Rightarrow c)$	$[(a \Rightarrow b) \wedge (b \Rightarrow c)] \Rightarrow (a \Rightarrow c)$
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	0	1	0	1
0	1	1	1	1	1	1	1
1	0	0	0	1	0	0	1
1	0	1	0	1	1	0	1
1	1	0	1	0	0	0	1
1	1	1	1	1	1	1	1

■

(*continued from page 37*)

negation. Thus, in the case of the distributivity of alternation over conjunction (considered above), if we write down the truth table for

$$\sim[a \vee (b \wedge c) \Leftrightarrow (a \vee b) \wedge (a \vee c)]$$

then the right-hand column of the table will consist entirely of 0's.

In *Unit 17, Logic II* we shall consider a number of different types of proof used in mathematics. Here we point out that, because we have been dealing with examples involving a *finite* number of propositions, we have been able to construct proofs by means of truth tables where every possible combination of truth values is taken into account. This method of proof, which exhausts all possible cases, is often known as proof by exhaustion. Of course, whilst it is theoretically possible to undertake proofs by exhaustion for all finite situations for which we can write down every possibility, as the number of possibilities increases the method becomes more and more exhausting! For this reason, we find it useful to find a more abstract approach to the propositional calculus and to develop an algebra in which theorems can be proved without recourse to the rather cumbersome method of truth tables, although this method is useful when only a few propositions are being considered.

Definition 3
*

Exercise 2

Exercise 2

Classify the following as a tautology, a contradiction, or neither:

(i) $a \Rightarrow (b \Rightarrow a)$
(ii) $\sim[(\sim a \Rightarrow \sim b) \Leftrightarrow (b \Rightarrow a)]$
(iii) $[(a \Leftrightarrow b) \Leftrightarrow (b \Leftrightarrow c)] \Leftrightarrow (a \Leftrightarrow c)$

■

11.1.8 Boolean Algebra

11.1.8

Main Text

* * *

We shall use this section to draw together some of the threads of the previous sections. In particular, we shall begin by summarising the relation between set algebra and the propositional calculus.

We saw in section 11.1.1 how negation and complementation are related, and in section 11.1.2 we saw how conjunction and alternation are related to intersection and union. We can therefore set up two columns headed "Propositional Calculus" and "Set Algebra" respectively, which summarise these relations as follows:

Propositional Calculus	Set Algebra
$\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots \in P$ (propositions)	$A, B, C, \ldots \subseteq U$ (sets)
unary operation $\sim$	unary operation $'$
binary operations $\wedge, \vee$	binary operations $\cap, \cup$
commutative laws $\mathbf{a} \wedge \mathbf{b} = \mathbf{b} \wedge \mathbf{a}$ $\mathbf{a} \vee \mathbf{b} = \mathbf{b} \vee \mathbf{a}$	commutative laws $A \cap B = B \cap A$ $A \cup B = B \cup A$
associative laws $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c}$ $\mathbf{a} \vee (\mathbf{b} \vee \mathbf{c}) = (\mathbf{a} \vee \mathbf{b}) \vee \mathbf{c}$	associative laws $A \cap (B \cap C) = (A \cap B) \cap C$ $A \cup (B \cup C) = (A \cup B) \cup C$
distributive laws $\mathbf{a} \wedge (\mathbf{b} \vee \mathbf{c})$ $= (\mathbf{a} \wedge \mathbf{b}) \vee (\mathbf{a} \wedge \mathbf{c})$ $\mathbf{a} \vee (\mathbf{b} \wedge \mathbf{c})$ $= (\mathbf{a} \vee \mathbf{b}) \wedge (\mathbf{a} \vee \mathbf{c})$	distributive laws $A \cap (B \cup C)$ $= (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C)$ $= (A \cup B) \cap (A \cup C)$
negation laws $\sim(\mathbf{a} \wedge \mathbf{b}) = \sim\mathbf{a} \vee \sim\mathbf{b}$ $\sim(\mathbf{a} \vee \mathbf{b}) = \sim\mathbf{a} \wedge \sim\mathbf{b}$	complementation laws $(A \cap B)' = A' \cup B'$ $(A \cup B)' = A' \cap B'$

Comparison of the two columns shows us that the two algebraic structures are identical. That is to say, if we map the elements of the set of propositions, P, one-one to the elements of a set of sub-sets of a universal set, U, we have an isomorphism:

$$m:(P, \sim, \wedge, \vee) \longmapsto (U, ', \cap, \cup),$$

and conversely. This means that, if a theorem is established in one of the algebras, then there will be a corresponding theorem in the other. For example, we have demonstrated by means of a Venn diagram that

$$A \cap (A' \cup B) = A \cap B$$

and that

$$A \cup (A' \cap B) = A \cup B$$

(*continued on page 40*)

Solution 11.1.7.2

Solution 11.1.7.2

(i) A tautology.
(ii) A contradiction.
(iii) A tautology. (Known as the *self-transitive law for equivalence.*) ■

(*continued from page 39*)

Because of the isomorphism, we should expect to find in the propositional calculus that

$$\mathbf{a} \wedge (\sim\mathbf{a} \vee \mathbf{b}) = \mathbf{a} \wedge \mathbf{b}$$

and

$$\mathbf{a} \vee (\sim\mathbf{a} \wedge \mathbf{b}) = \mathbf{a} \vee \mathbf{b}.$$

Reference to appropriate truth tables will confirm these conjectures.

(In thinking about the isomorphism

$$m:(P, \sim, \wedge, \vee) \longmapsto (U, ', \cap, \cup),$$

you may have noticed that we have avoided detailed discussion of the actual mapping between the set of propositions, P, and the universal set, U. We admit that there are certain difficulties here in that, whilst U *is itself a set*, P as a collection of propositions is not itself a proposition. In fact, we do not want to become involved in this particular issue just at this point, but we do hope to throw some light on the matter after we have introduced Boolean Algebra, of which both set algebra and the propositional calculus are examples.)

The algebraic structure of the algebra of sets and the propositional calculus is known as a Boolean Algebra. This algebra can bé set up in an entirely abstract manner without reference to any particular interpretation such as the propositional calculus or set algebra. We shall now do just this, as follows:

We consider a set of elements:

$$B = \{x, y, z, \ldots\}$$

We define two binary operations on B,

Definition 1
* * *

$\cap$ (cap),

$\cup$ (cup),

with the following properties:

B1 $\quad x \cap y = y \cap x$
B2 $\quad x \cup y = y \cup x$ } commutative properties

B3 $\quad x \cap (y \cap z) = (x \cap y) \cap z$
B4 $\quad x \cup (y \cup z) = (x \cup y) \cup z$ } associative properties

B5 $\quad x \cap (x \cup y) = x$
B6 $\quad x \cup (x \cap y) = x$ } absorptive properties

B7 $\quad x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$
B8 $\quad x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$ } distributive properties

We define two elements of B, O and I, satisfying:

B9 $\quad x \cap O = O$
B10 $\quad x \cap I = x$
B11 $\quad x \cup O = x$
B12 $\quad x \cup I = I$ } identity properties

We define a unary operation (complement) on B by introducing an element $x' \in B$ corresponding to each element x and satisfying:

B13 $\quad x \cap x' = O$

B14 $\quad x \cup x' = I$

There are a number of things which you have probably already noticed but to which we should like to draw your attention particularly. First, we now read "$x \cap y$" and "$x \cup y$" as "x cap y" and "x cup y" respectively, and not as "x intersection y" and "x union y" as in set algebra. Secondly, we have included two laws, which we call absorption laws, in our definition of a Boolean algebra; you may care to check for yourself that these correspond to equivalent situations both in set algebra and in the propositional calculus. Thirdly, we have defined I and O to be identity elements for cap and cup respectively. (Properties B10 and B11 are similar to the properties of the identity elements 1 and 0 for multiplication and addition respectively in ordinary arithmetic:

Discussion
* *

Definition 2
*

$a \times 1 = a \qquad$ for all $a \in R$

and

$a + 0 = a \qquad$ for all $a \in R$.)

In set algebra, I is the universal set, U, and O is the empty (or null) set, $\varnothing$, and we have:

$A \cap U = A \qquad$ for all $A \subseteq U$

and

$A \cup \varnothing = A \qquad$ for all $A \subseteq U$.

In the propositional calculus, we can think of the contradiction

$\mathbf{a} \wedge \sim\mathbf{a}$

giving us "universal falsehood" and the tautology

$\mathbf{a} \vee \sim\mathbf{a}$

giving us "universal truth". If we denote these by **F** and **T** respectively, then we can extend our two columns on page 39 as follows:

Propositional Calculus	Set Algebra
$\mathbf{a} \wedge \mathbf{F} = \mathbf{F}$	$A \cap \varnothing = \varnothing$
$\mathbf{a} \wedge \mathbf{T} = \mathbf{a}$	$A \cap U = A$
$\mathbf{a} \vee \mathbf{F} = \mathbf{a}$	$A \cup \varnothing = A$
$\mathbf{a} \vee \mathbf{T} = \mathbf{T}$	$A \cup U = U$
$\mathbf{a} \wedge \sim\mathbf{a} = \mathbf{F}$	$A \cap A' = \varnothing$
$\mathbf{a} \vee \sim\mathbf{a} = \mathbf{T}$	$A \cup A' = U$

It is because of the special status of U that it appears *both* as the image of P under the morphism

$$m:(P, \sim, \wedge, \vee) \longmapsto (U, ', \cap, \cup)$$

and also as the identity element of intersection in the algebra of sets.

We can now prove theorems without reference to Venn diagrams or truth tables entirely by algebraic manipulation, and any theorem so proved will be applicable both to the propositional calculus and to set algebra. For example, we can prove that

$$(x \cap y)' = x' \cup y'$$

as follows (at the end of each line we indicate the properties used).

We want to show that $x' \cup y'$ is equal to the complement of $x \cap y$. Looking at properties B13 and B14 and noting that the complement is unique, we shall prove that:

$$(x \cap y) \cap (x' \cup y') = O \qquad \text{(A)}$$

$$(x \cap y) \cup (x' \cup y') = I \qquad \text{(B)}$$

Part (A)

$$\begin{aligned}
&(x \cap y) \cap (x' \cup y') \\
&= [(x \cap y) \cap x'] \cup [(x \cap y) \cap y'] && \text{(B7)} \\
&= [(x' \cap x) \cap y] \cup [(y' \cap y) \cap x] && \text{(B3, B1)} \\
&= (O \cap y) \cup (O \cap x) && \text{(B13, B1)} \\
&= O \cup O && \text{(B9, B1)} \\
&= O && \text{(B11)}
\end{aligned}$$

Part (B)

$$\begin{aligned}
&(x \cap y) \cup (x' \cup y') \\
&= (x' \cup y') \cup (x \cap y) && \text{(B2)} \\
&= [(x' \cup y') \cup x] \cap [(x' \cup y') \cup y] && \text{(B8)} \\
&= [(x' \cup x) \cup y'] \cap [x' \cup (y' \cup y)] && \text{(B4, B2)} \\
&= (I \cup y') \cap (x' \cup I) && \text{(B14, B2)} \\
&= I \cap I && \text{(B12, B2)} \\
&= I && \text{(B10)}
\end{aligned}$$

Since (A) and (B) are satisfied, we have proved the required result.

Exercise 1

Exercise 1
(3 minutes)

If x, y are elements of a Boolean algebra, prove that

$$x \cap (x' \cup y) = x \cap y$$

$(x \cap x') \cup (x \cap y)$

$O \cup (x \cap y)$

$x \cap y$

11.2 SWITCHING SYSTEMS

11.2

11.2.0 Introduction

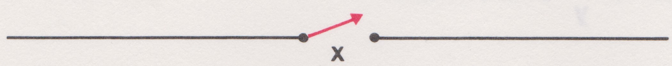
11.2.0

Introduction
* *

In section 11.1 we discussed the propositional calculus and demonstrated that its structure is isomorphic to that of the algebra of sets. We ended the section by introducing the abstract algebra, known as *Boolean algebra*, which underlies both propositions and sets. We shall now show how this same algebra also underlies the ON/OFF systems which we call switching systems.

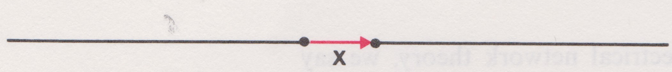
When you switch on the light in a room you expect a lamp or tube to light up, and when you switch off the light you expect the same lamp or tube to go out.

We can represent this ON/OFF situation by the diagram:



in which x represents the light switch. In this diagram x is shown in the condition such that "flow" along the line is *prevented*, and we shall say that x is OFF and that the path is OPEN. When x is in the condition such that "flow" along the line is *permitted* we have the situation of the diagram:

Definition 1
* * *



and we shall say that x is ON and that the path is CLOSED. This may seem somewhat different from the more common use of "OPEN" and "CLOSED" in ordinary English, but the convention has arisen from the meanings given to these words in electrical network theory, though such "flow" situations arise in many situations other than electrical ones.

Definition 2
* * *

We can compare the two-state device, which we call a switch, with a proposition. A two-state device is a device by means of which some path is made *either* OPEN *or* CLOSED. A proposition is a statement which is *either* FALSE *or* TRUE. Similarly, given a set and an object, *either* the object is a member of the set *or* it is not.

Definition 3
* *

We now go on to consider how we can combine paths which incorporate switches, and how such combinations lead us to an algebra of switching which is an example of the Boolean algebra of the last sub-section, and is hence isomorphic to the calculus of propositions and to the algebra of sets. We end this text with a final example, showing how a pictorial method of carrying out an algebraic operation of special relevance in switching system design can also be of interest when interpreted in terms of the propositional calculus and of set theory.

Solution 11.1.8.1 **Solution 11.1.8.1**

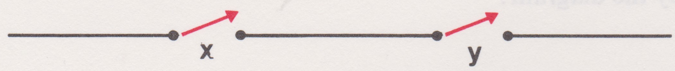
$$
\begin{aligned}
x \cap (x' \cup y) &= (x \cap x') \cup (x \cap y) && \text{(B7)}\\
&= O \cup (x \cap y) && \text{(B13)}\\
&= (x \cap y) && \text{(B11, B2)} \quad \blacksquare
\end{aligned}
$$

11.2.1 Switching Algebra

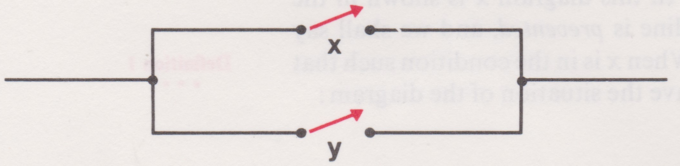
11.2.1

Main Text * *

We start by considering the two ways in which we can arrange a pair of simple ON/OFF switches, *x*, *y*. They may be arranged sequentially, as here:



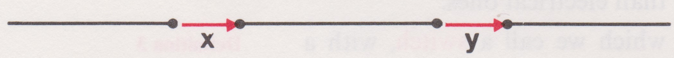
or they may be arranged side by side, as here:



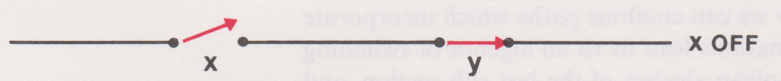
Following again the terminology of electrical network theory, we say in the former case that *x* and *y* are in series, and in the latter case that they are in parallel.

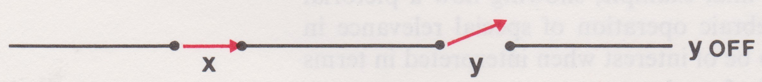
Definition 1 * * *

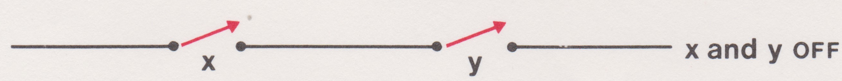
Let us look first at the case where *x* and *y* are in series. Here, flow is permitted right along the path, i.e. the path is CLOSED, *only if both x and y are* ON giving us the situation:



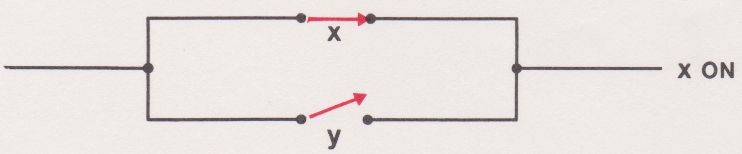
If one or both of *x* or *y* is OFF then flow will be prevented, i.e. the path will be OPEN, thus:

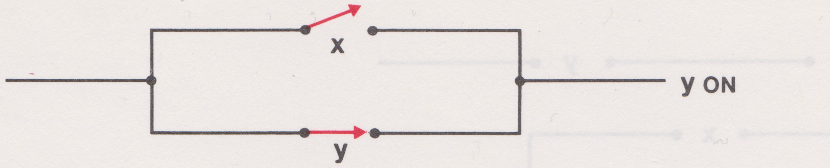


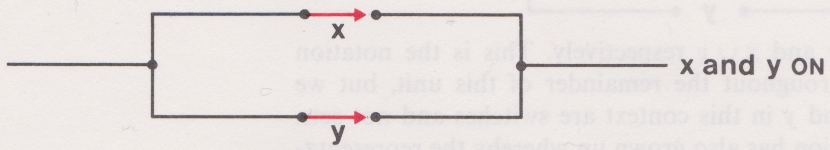




Looking now at the case when *x* and *y* are in parallel, we find that flow will be permitted right along the path *provided that one at least of x and y is* ON. The three possible cases are:







We now compare these two arrangements of switches x and y with the combination of two propositions under conjunction and alternation.

In switching theory it is customary to give x the value 1 when switch x is ON so that flow is permitted, and the value 0 when switch x is OFF so that flow is prevented. This corresponds to giving a proposition the truth value 1 when it is assumed TRUE and 0 when it is assumed FALSE.

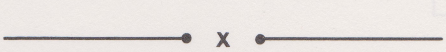
In the case in which x and y are in series *both* x and y have to be ON for flow to be permitted, so that we have a situation corresponding to the conjunction of two propositions **a**, **b**, where both have to be TRUE for the compound proposition $\mathbf{a} \wedge \mathbf{b}$ to be TRUE also. The truth table for conjunction can thus be interpreted in switching theory as:

Switch x	Switch y	Switches x, y in Series
0 (OFF)	0 (OFF)	0 (flow prevented)
0 (OFF)	1 (ON)	0 (flow prevented)
1 (ON)	0 (OFF)	0 (flow prevented)
1 (ON)	1 (ON)	1 (flow permitted)

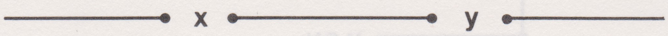
In the case in which x and y are in parallel, *either x or y* (or both) has to be ON for flow to be permitted, so that the situation corresponds to the alternation of two propositions where the compound proposition $\mathbf{a} \vee \mathbf{b}$ is TRUE provided that one at least of **a**, **b** is TRUE. The truth table for alternation can thus be interpreted as:

Switch x	Switch y	Switches x, y in Parallel
0 (OFF)	0 (OFF)	0 (flow prevented)
0 (OFF)	1 (ON)	1 (flow permitted)
1 (ON)	0 (OFF)	1 (flow permitted)
1 (ON)	1 (ON)	1 (flow permitted)

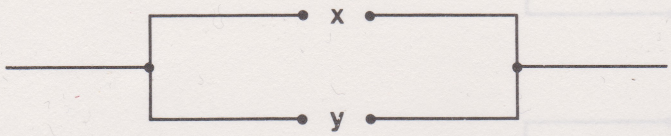
Because the switching algebra structure is isomorphic to that of the propositional calculus and hence also to that of set algebra, we adopt the symbols from Boolean algebra, namely $\cap$ and $\cup$, to denote series and parallel connections respectively. If we represent a switch x which may be *either* ON *or* OFF by:



then the networks:



and:



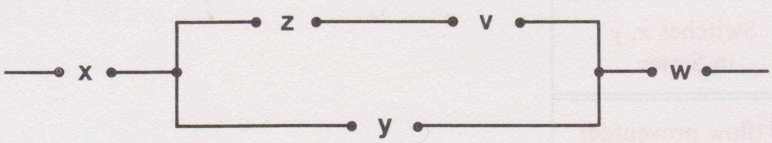
are represented by $x \cap y$ and $x \cup y$ respectively. This is the notation which we shall adopt throughout the remainder of this unit, but we must remember that x and y in this context are switches and not sets. (Unfortunately, a convention has also grown up whereby the representation of the series and parallel connection of switches x and y can also be xy and $x + y$. One of the objections to this notation is that multiplication and addition are not each distributive over the other, and when we have two operations which are mutually distributive it is better to adopt symbols which in a corresponding algebra, in this case Boolean algebra, do represent operations having this property.) Because of the analogy with conjunction and alternation, the circuit which corresponds to the series connection of two switches is often called an AND-gate and that corresponding to the parallel connection of two switches is often called an OR-gate.

Definition 2

Exercise 1

Exercise 1
(3 minutes)

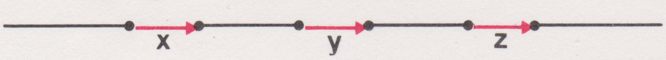
Write down an algebraic representation for the following switching system:



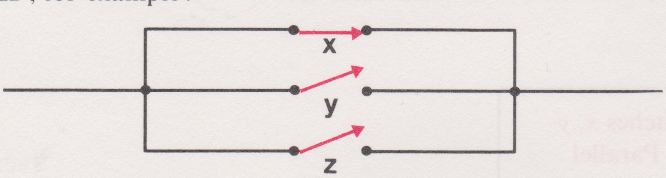
■

Main Text

If we now extend our AND- and OR-gates to include three switches, for the AND-gate all three switches must be CLOSED to permit flow; i.e. we get a CLOSED circuit only in the case:



and for the OR-gate flow is permitted when *any one of the three switches* is CLOSED; for example:



The corresponding tables are:

x	y	z	$x \cap y \cap z$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

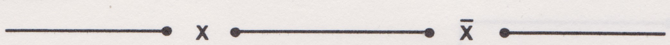
and

x	y	z	$x \cup y \cup z$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

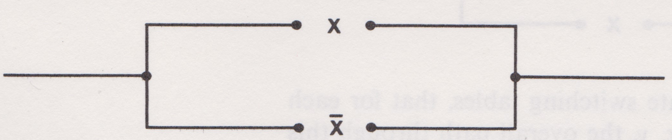
These two tables correspond to the truth tables for the conjunction and alternation respectively of three propositions in the propositional calculus. (Notice that we have omitted the brackets in the three-term expressions, because the isomorphism with the propositional calculus assures us that both $\cap$ and $\cup$ are associative.)

Discussion
* *

We may now ask what it is in switching algebra that corresponds to the logical operation of negation. In answering this question we should notice that the negation of the truth value 1 is 0, and that the negation of 0 is 1. Thus, in switching algebra, the "opposite" of a CLOSED path is an OPEN path and the "opposite" of an OPEN path is a CLOSED path. If then, x represents a switch and **a** represents a proposition, we have corresponding to $\sim$**a** (which is FALSE when **a** is TRUE and TRUE when **a** is FALSE) a switch, $\bar{x}$ say, which is OFF when x is ON and ON when x is OFF. We then have



is always OPEN, whatever the state of x, and



Definition 3
* *

is always CLOSED, whatever the state of x. If we call a CLOSED path the complementary path to an OPEN path and vice versa, these results can be stated as

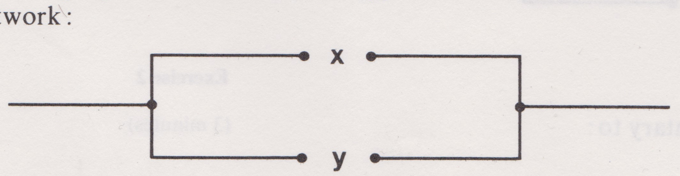
"a path in series with its complementary path is an OPEN path"

and

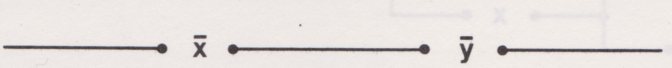
"a path in parallel with its complementary path is a CLOSED path"

respectively.

We can now try to find the switching system (or network) which is complementary to a given network by appealing to the corresponding situation in the propositional calculus. Suppose, for instance, that we have the network:



i.e. a simple OR-gate. In logic, the corresponding situation is the alternation of propositions **a** and **b**. The negation of **a** $\vee$ **b** is the conjunction of $\sim$**a** and $\sim$**b**, so, by analogy, the complementary network which we are seeking will be the series connection of $\bar{x}$ and $\bar{y}$:



(*continued on page 48*)

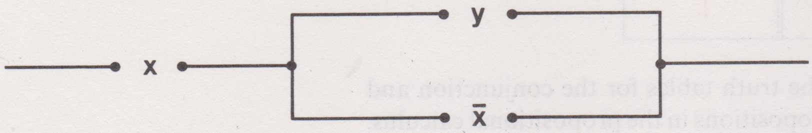
Solution 1 **Solution 1**

$$x \cap [(z \cap v) \cup y] \cap w,$$

or an equivalent form.

(continued from page 47)

As a second example, consider the network represented by:



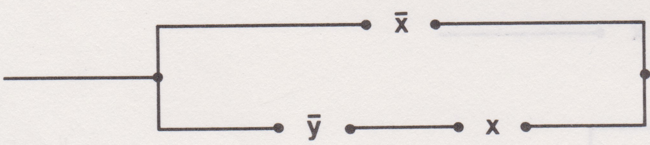
This is a switch x in series with the parallel connection of y and $\bar{x}$ (the switch complementary to x). If we seek the complementary network we can again look to the propositional calculus which tells us that the negation of

$$\mathbf{a} \wedge (\mathbf{b} \vee \sim\mathbf{a})$$

is

$$\sim\mathbf{a} \vee (\sim\mathbf{b} \wedge \mathbf{a}).$$

Translating back to switching terms, we arrive at the network:

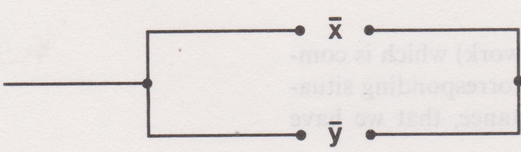


You should check, by using appropriate switching tables, that for each possible combination of the states of x, y, the overall path through this latter network is OPEN whenever the path through the former is CLOSED and vice versa. We can, however, pursue this example further. From the propositional calculus we know that

$$\mathbf{a} \wedge (\mathbf{b} \vee \sim\mathbf{a}) = \mathbf{a} \wedge \mathbf{b},$$

the negation of $\mathbf{a} \wedge \mathbf{b}$ being $\sim\mathbf{a} \vee \sim\mathbf{b}$.

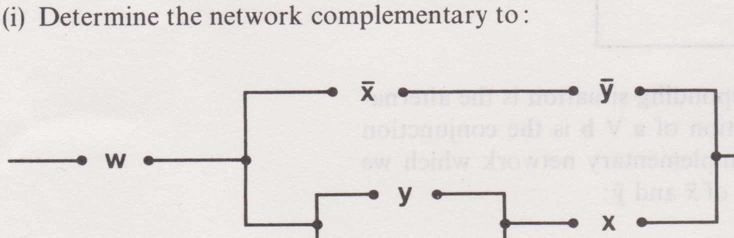
Thus the complementary network which we derived must be equivalent to:



Exercise 2 **Exercise 2**

(3 minutes)

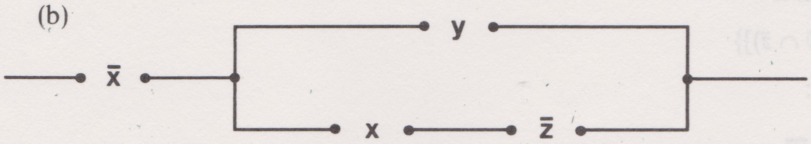
(i) Determine the network complementary to:



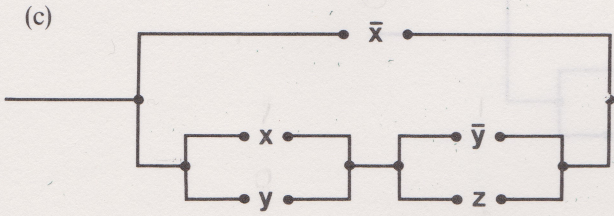
(ii) Determine for which of the values 0, 1 of x, y, z, flow through the following networks will be permitted: (2 minutes)

(a)

(b)



(c)



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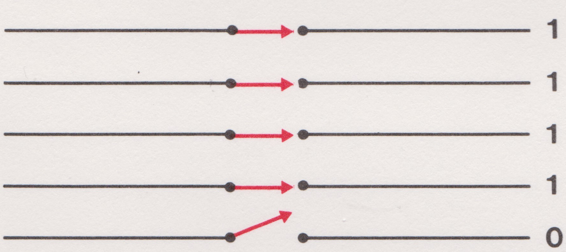
Application

One of the most important developments of switching theory has been the design of digital computers. Most of these computers carry out their operations by means of binary arithmetic, in which only the digits 0 and 1 are used, and thus numbers and coded instructions for operations upon numbers are conveniently represented by electrical or electronic switches. The number 30, for example, becomes 11110 in the binary system because

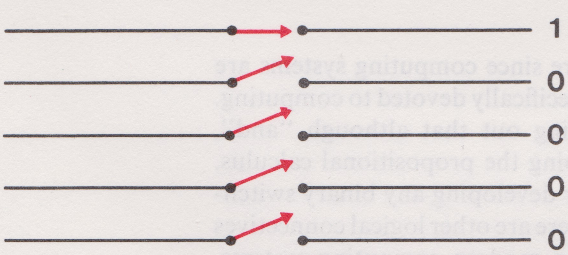
(See RB12)

$$30 = 1{\cdot}2^4 + 1{\cdot}2^3 + 1{\cdot}2^2 + 1{\cdot}2^1 + 0{\cdot}2^0$$

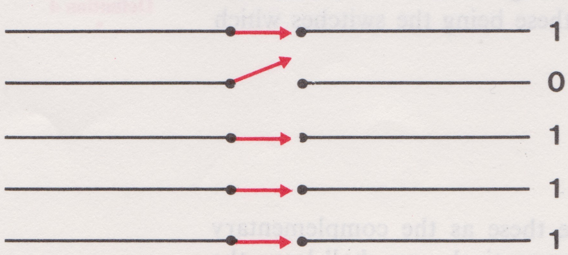
We can thus represent 30 by a switching system with five paths, through four of which flow is permitted, thus:



The number 16 would be represented by:



since $16 = 1{\cdot}2^4$, and the number 23 by:



since $23 = 1{\cdot}2^4 + 0{\cdot}2^3 + 1{\cdot}2^2 + 1{\cdot}2^1 + 1{\cdot}2^0$

(continued on page 50)

Solution 2 **Solution 2**

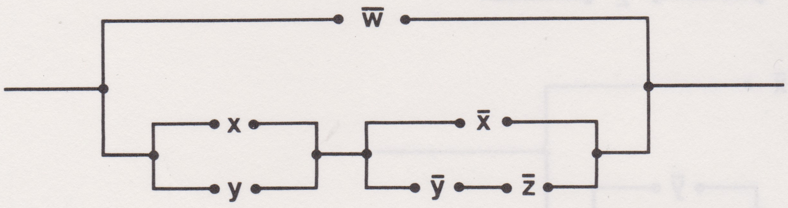
(i) The switching expression corresponding to the given network is

$$w \cap \{(\bar{x} \cap \bar{y}) \cup [x \cap (y \cup z)]\}$$

The complementary expression is

$$\bar{w} \cup \{(x \cup y) \cap [\bar{x} \cup (\bar{y} \cap \bar{z})]\}$$

which represents the network:



(ii) (a) $x = 1$ *and* $y = 0$ (z is not in the network)

(b) $x = 0$ *and* $y = 1$

(There can never be flow along the bottom path since this would require $x = 0$ and $x = 1$ simultaneously. The value of z therefore does not affect flow through the network.)

(c) $x = 0$ (whatever the values of y, z)
or $x = 1$ *and* $y = 0$ (whatever the value of z)
or $x = 1$ *and* $z = 1$ (whatever the value of y)
or $y = 1$ *and* $z = 1$ (whatever the value of x). ■

(*continued from page 49*)

If we letter the switches in each of the above diagrams from top to bottom with the letters v, w, x, y, z, we see that 30 requires that

$$\left.\begin{array}{r} v = w = x = y = 1 \\ z = 0 \end{array}\right\},$$

16 requires that

$$\left.\begin{array}{r} v = 1 \\ w = x = y = z = 0 \end{array}\right\},$$

and 23 requires that

$$\left.\begin{array}{r} v = x = y = z = 1 \\ w = 0 \end{array}\right\}$$

We shall not pursue this topic further here since computing systems are discussed in a part of the course which is specifically devoted to computing. We shall end this sub-section by pointing out that although "and", "or" and "not" are sufficient for developing the propositional calculus, and hence the corresponding networks for developing any binary switching system which we may wish to design, there are other logical connectives whose switching counterparts are used in modern computing systems. Two such are $\downarrow$ and $|$, and the corresponding switches are known as a NOR-gate and a NAND-gate respectively, these being the switches which perform operations analogous to:

Definition 4
*

"neither **a** nor **b**"

and

"not both **a** and **b**"

respectively in logic. You will recognize these as the complementary circuits to an OR-gate and an AND-gate respectively; we shall leave the construction of their switching tables to the exercise which follows. (In

the television programme, we look at a simple application of the NOR-gate principle in the circuitry which controls the response of a lift when you push a button on a landing calling it to your floor.)

Exercise 3

Exercise 3
(4 minutes)

Write down the switching table corresponding to

(i) a NOR-gate
(ii) a NAND-gate.

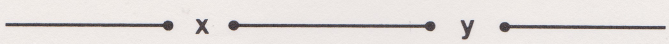
11.2.2 Networks and Logic Diagrams

11.2.2

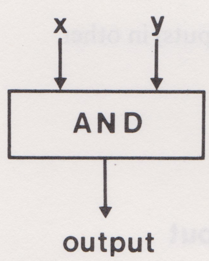
Notation
* *

We shall not introduce any new theory in this section but shall simply consider an alternative diagrammatic representation of switching circuits which enables us to relate a system to its basic logical components easily.

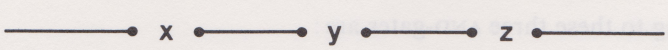
We start by considering again the basic idea of an AND-gate. For two switches in series, we saw that the network is:



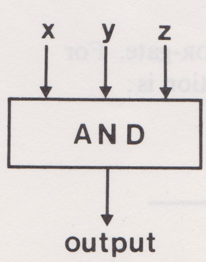
An alternative representation is:



Here we have an output only if both *x and y* are providing an input. This kind of picture, known as a *logic diagram*, is very frequently used by system designers. For three switches arranged as an AND-gate we have:



and the corresponding logic diagram:



(*continued on page 52*)

Solution 3

Solution 3

(i)

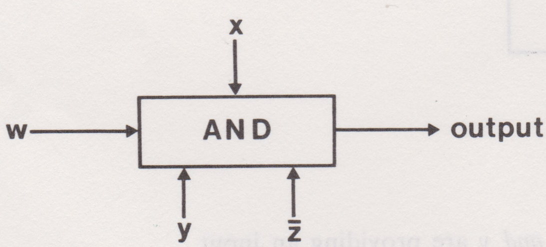
x	y	Overall Flow (NOR-gate)
0	0	1
0	1	0
1	0	0
1	1	0

(ii)

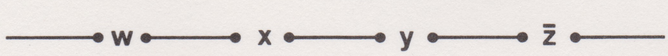
x	y	Overall Flow (NAND-gate)
0	0	1
0	1	1
1	0	1
1	1	0

(continued from page 51)

We can arrange the inputs to the gate and the output (or outputs) in other ways. For example, the logic diagram:



corresponds to the network:



The algebraic expressions corresponding to these three AND-gates are:

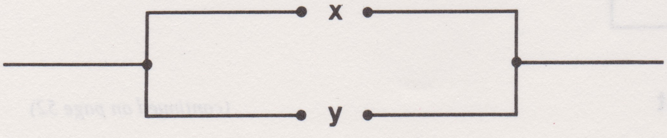
$x \cap y$

$x \cap y \cap z,$

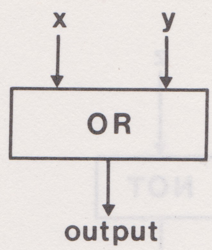
$w \cap x \cap y \cap \bar{z}$

respectively.

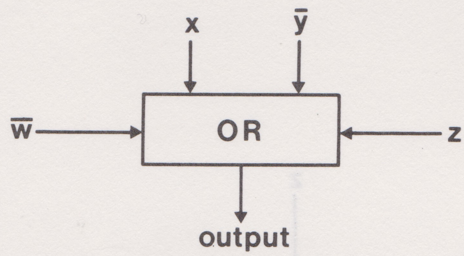
We can adopt a similar pictorial representation for the OR-gate. For two switches in parallel, we saw that the network representation is:



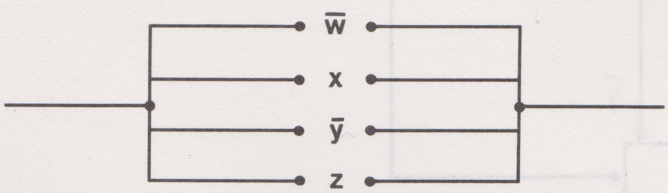
The alternative representation as a logic diagram is:



Here we have an output if *either* x *or* y *or* both are providing input. For several inputs to an OR-gate we have, for example, the logic diagram:



representing the network:



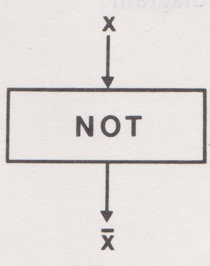
The algebraic expressions corresponding to these two OR-gates are

$x \cup y$,

$\bar{w} \cup x \cup \bar{y} \cup z$,

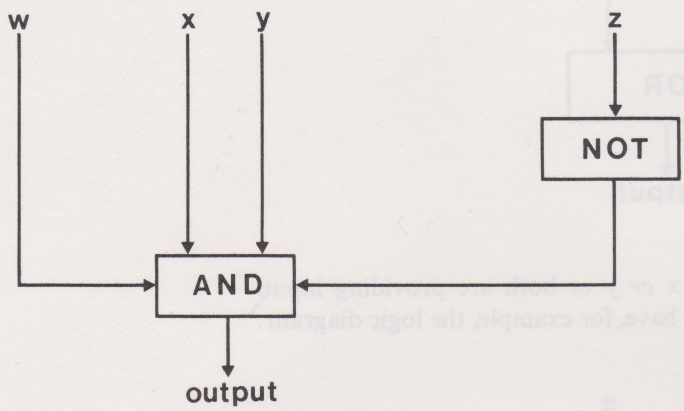
respectively.

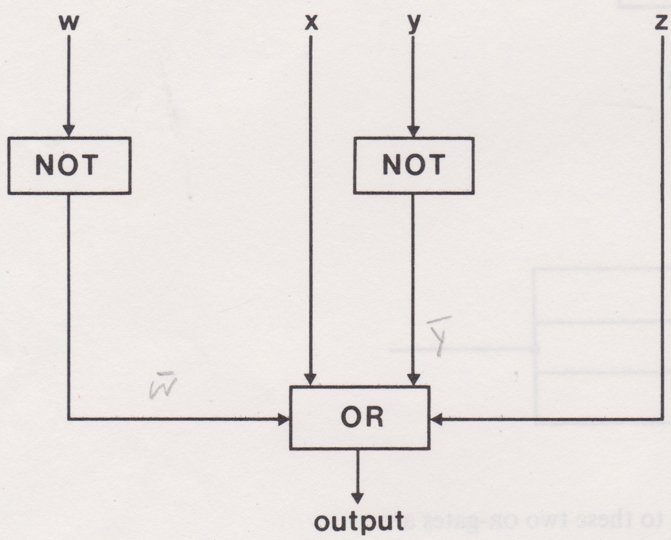
A similar logic diagram is also used for the gate whose output is the complement of its input, i.e. the NOT-gate; we thus have: **Definition 1**



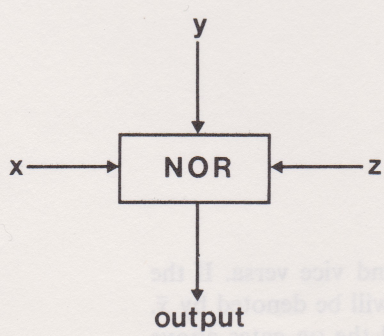
Here we have no output if x is providing input, and vice versa. If the input to this gate is denoted by x, then the output will be denoted by $\bar{x}$. We can now represent the circuits of the AND- and the OR-gates above

(with corresponding algebraic expressions $w \cap x \cap y \cap \bar{z}$ and $\bar{w} \cup x \cup \bar{y} \cup z$ respectively) in the alternative forms:





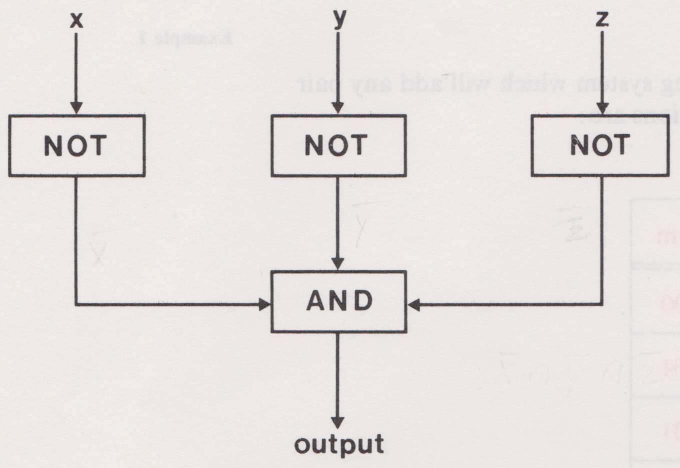
In addition to AND-, OR- and NOT-gates, other logical gates can be represented in the same way. We saw at the end of the last section that two other gates which are used by system designers are the NOR-gate and the NAND-gate. These last two gates enable us to provide alternative representations of logic networks since they effectively combine the basic "and" and "or" operations with complementation. Thus the logic diagram:



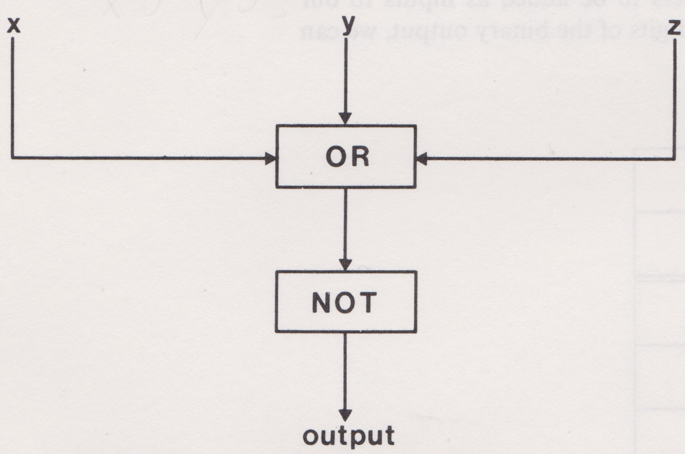
which has as its corresponding algebraic expression

$$\bar{x} \cap \bar{y} \cap \bar{z}$$

is an alternative to:



and also to:



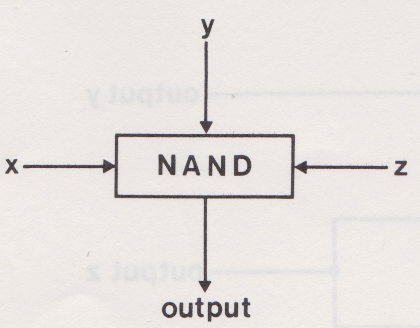
the latter having as its corresponding algebraic expression:

$$\overline{x \cup y \cup z}$$

which is algebraically equivalent to:

$$\bar{x} \cap \bar{y} \cap \bar{z}.$$

The logic diagram:



also provides alternative representation of diagrams using only AND-, OR- and NOT-gates; you will be asked to sketch these in the next exercise.

We conclude this text by considering a very simple example of switching system design.

Example 1

Example 1

Let us suppose that we want a switching system which will add any pair of the numbers 0, 1. The possible situations are:

Decimal Form	Binary Form
$0 + 0 = 0$	$0 + 0 = 00$
$0 + 1 = 1$	$0 + 1 = 01$
$1 + 0 = 1$	$1 + 0 = 01$
$1 + 1 = 2$	$1 + 1 = 10$

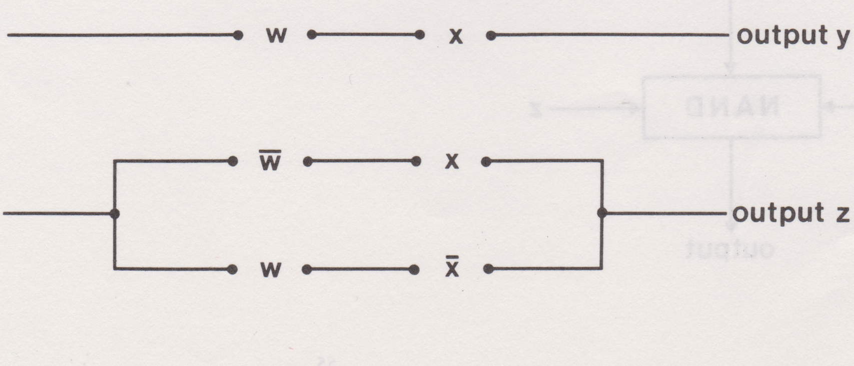
If we let w and x represent the numbers to be added as inputs to our system, and y and z represent the two digits of the binary output, we can draw up the table:

Input		Output	
w	x	y	z
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

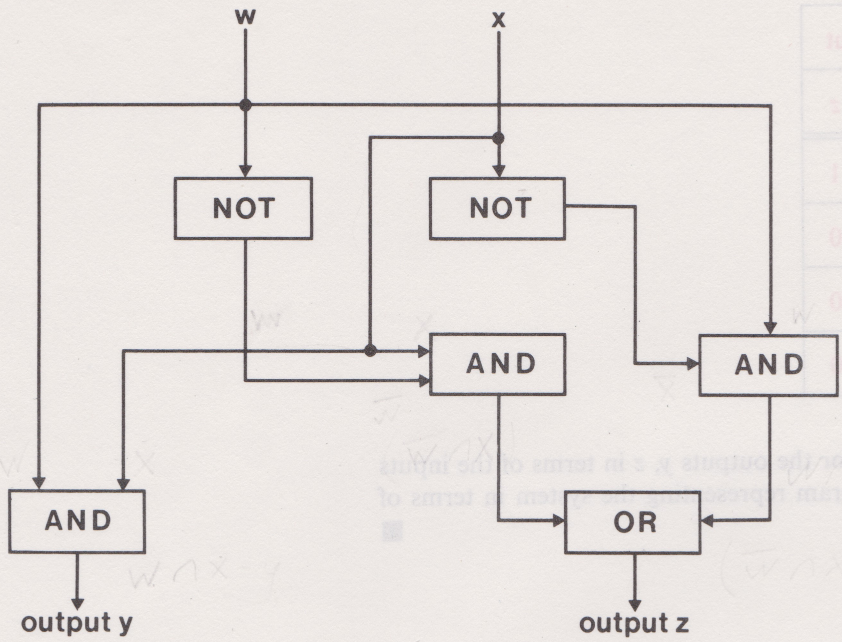
Considering the output y only, we can see that we have exactly the switching table for an AND-gate (see page 45); thus the appropriate algebraic representation is $y = w \cap x$.

Considering z only, we have the "exclusive-or" situation and the corresponding algebraic representation is $z = (\overline{w} \cap x) \cup (w \cap \overline{x})$.

We can now represent the required system in network form as:



and in logic diagram form as:

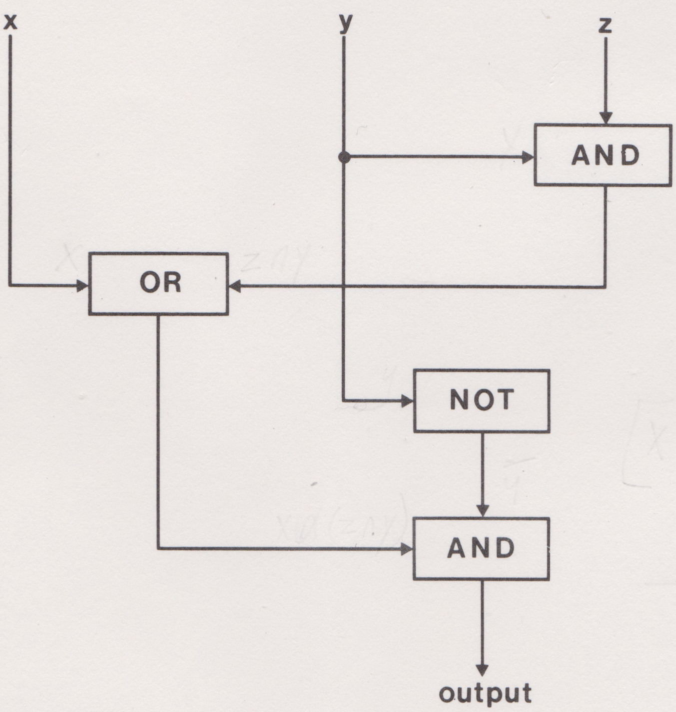


(Notice the convention of using "blobs" to indicate where paths divide or intersect.) ■

Exercise 1

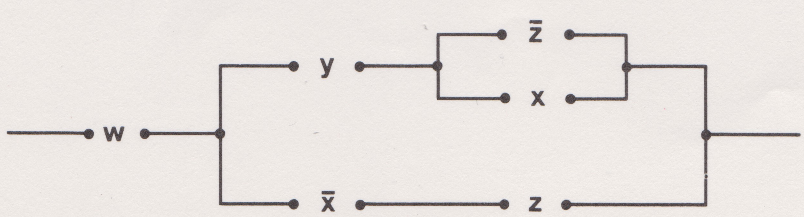
Exercise 1

(i) (a) Sketch a network which corresponds to the logic diagram: (2 minutes)



$[x \cup (z \cap y)] \cap \bar{y}$

(b) Sketch a logic diagram which corresponds to the network: (2 minutes)



(ii) A simple switching system is represented by the following table:

Input		Output	
w	x	y	z
0	0	1	1
0	1	1	0
1	0	1	0
1	1	0	0

Give algebraic expressions for the outputs y, z in terms of the inputs w, x and devise a logic diagram representing the system in terms of NAND- and NOR-gates only. ■

11.3 CONCLUSION

11.3

Conclusion

In this text we have tried to show that Boolean Algebra is an example of a mathematical structure, the recognition of which can unify different branches of mathematics, which in this case are set algebra, propositional calculus, and switching networks. We have confined our attention to simple examples involving only a very few sets, propositions or switches. There need be no new principle introduced, however, in extending this work to cover examples in which a very large number of sets, propositions or switches are involved as, for example, in the complicated switching system of a large digital computer. Because network components are expensive and may take up valuable space, it is important to obtain the simplest form of an expression corresponding to a particular network. It is therefore important to be able to manipulate the algebra correctly.

Postscript

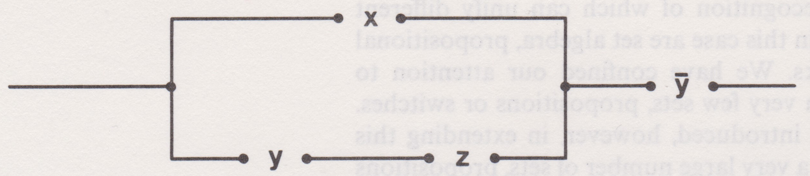
" 'Contrariwise,' continued Tweedledee, 'if it was so, it might be; and if it were so, it would be: but as it isn't, it ain't. That's logic.' "

Lewis Carroll

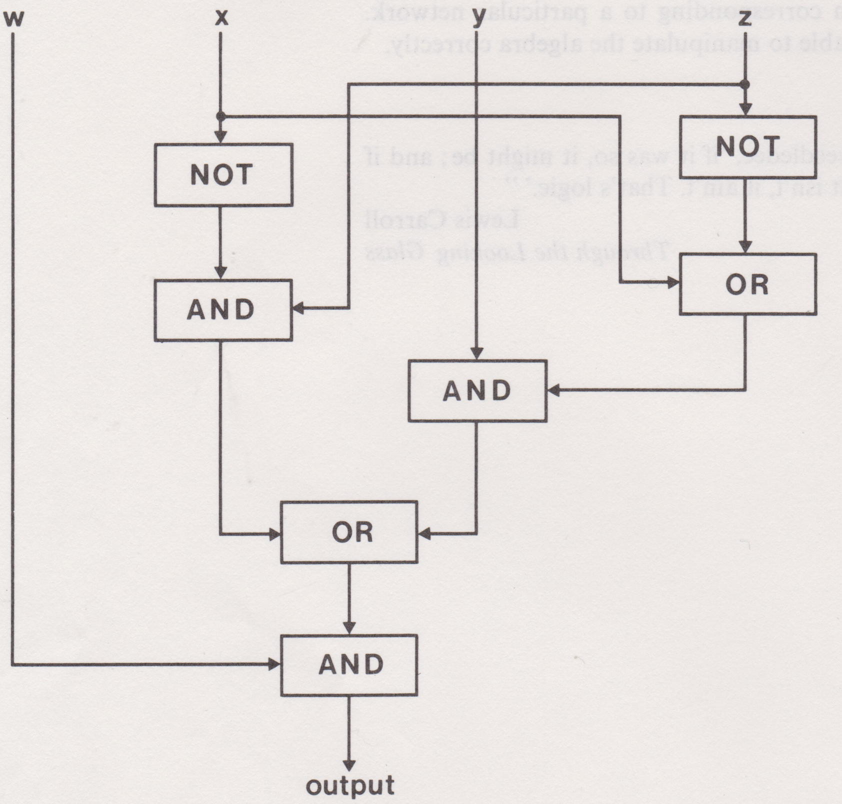
Through the Looking Glass

Solution 11.2.2.1

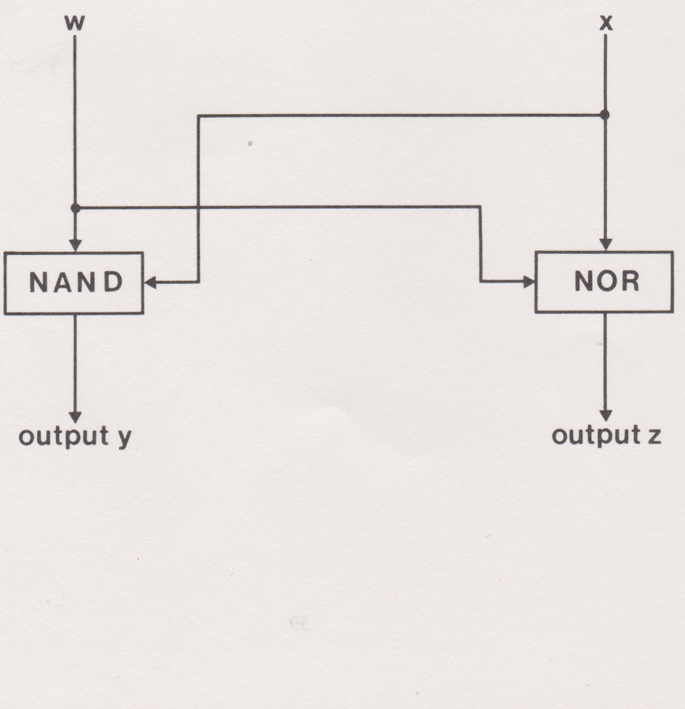
(i) (a) One possible answer is:



(b) One possible answer is:



(ii) $y = \overline{w \cap x}$ (or equivalent)
$z = \bar{w} \cap \bar{x}$ (or equivalent)
One possible answer is:

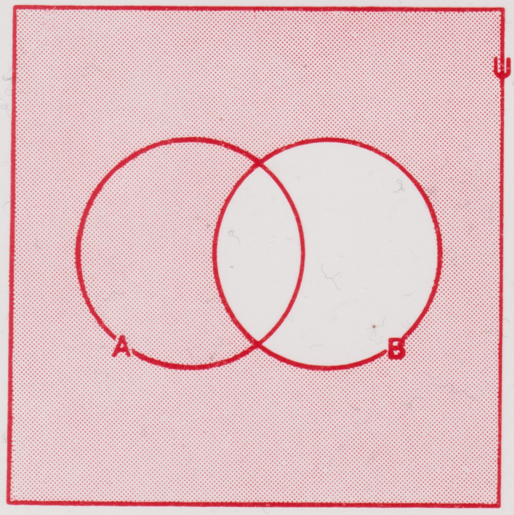


■

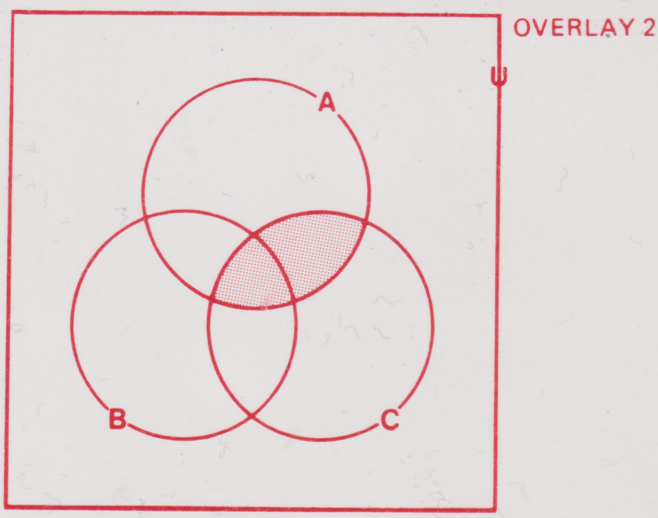
M100—MATHEMATICS FOUNDATION COURSE UNITS

1 Functions
2 Errors and Accuracy
3 Operations and Morphisms
4 Finite Differences
5 NO TEXT
6 Inequalities
7 Sequences and Limits I
8 Computing I
9 Integration I
10 NO TEXT
11 Logic I—Boolean Algebra
12 Differentiation I
13 Integration II
14 Sequences and Limits II
15 Differentiation II
16 Probability and Statistics I
17 Logic II—Proof
18 Probability and Statistics II
19 Relations
20 Computing II
21 Probability and Statistics III
22 Linear Algebra I
23 Linear Algebra II
24 Differential Equations I
25 NO TEXT
26 Linear Algebra III
27 Complex Numbers I
28 Linear Algebra IV
29 Complex Numbers II
30 Groups I
31 Differential Equations II
32 NO TEXT
33 Groups II
34 Number Systems
35 Topology
36 Mathematical Structures

OVERLAY 1

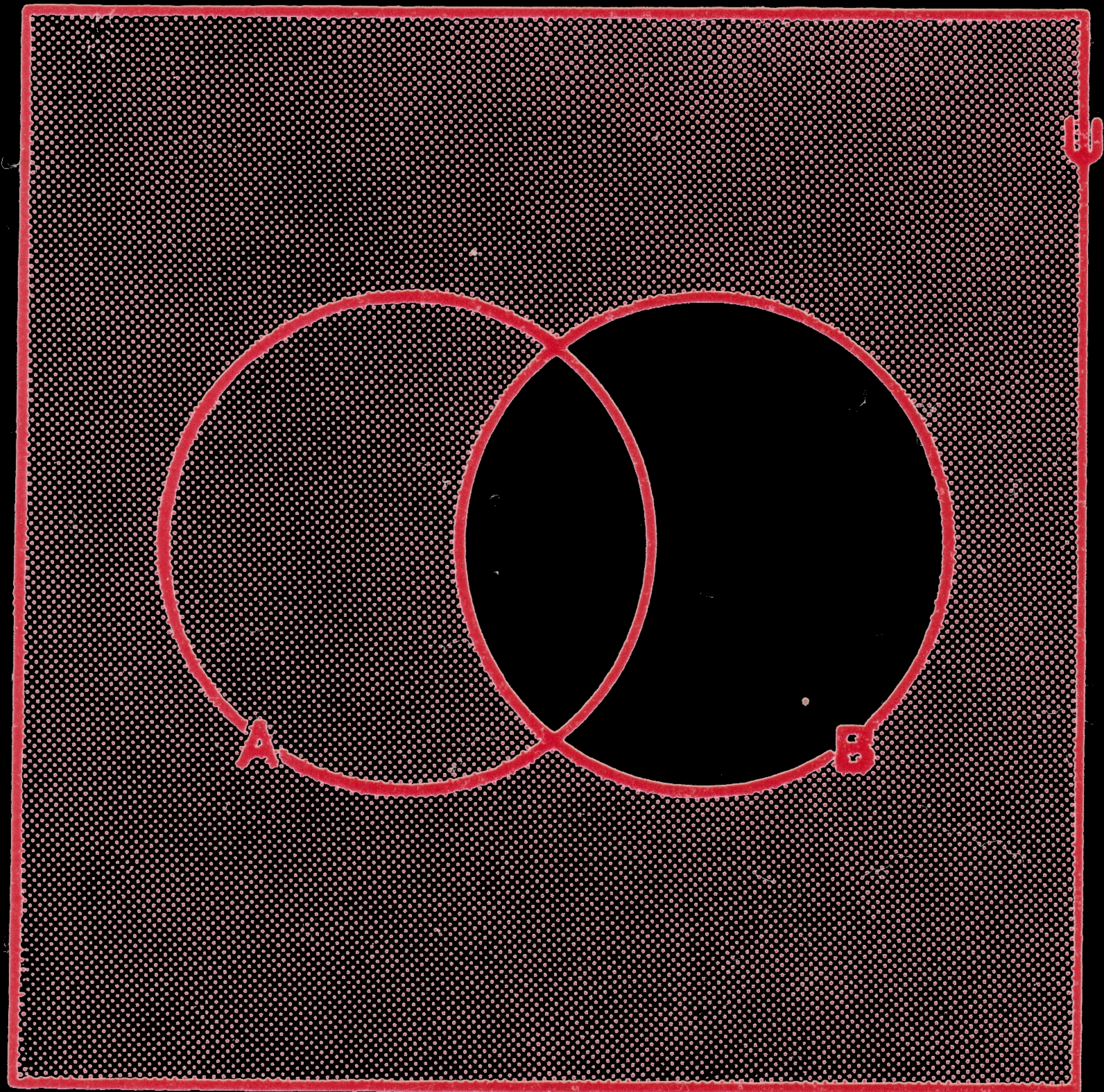


B′ shown shaded

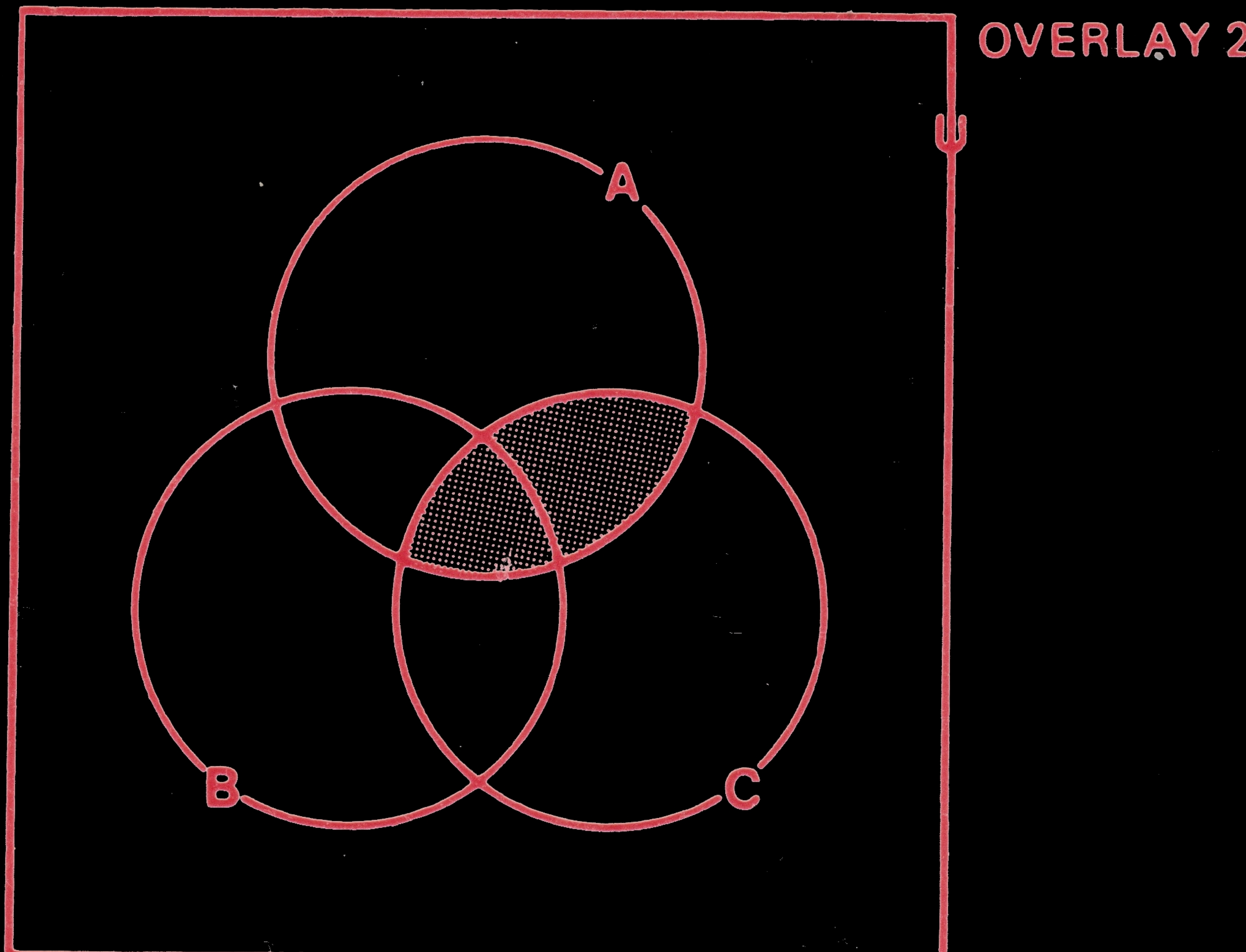


A ∩ C shaded

OVERLAY 1



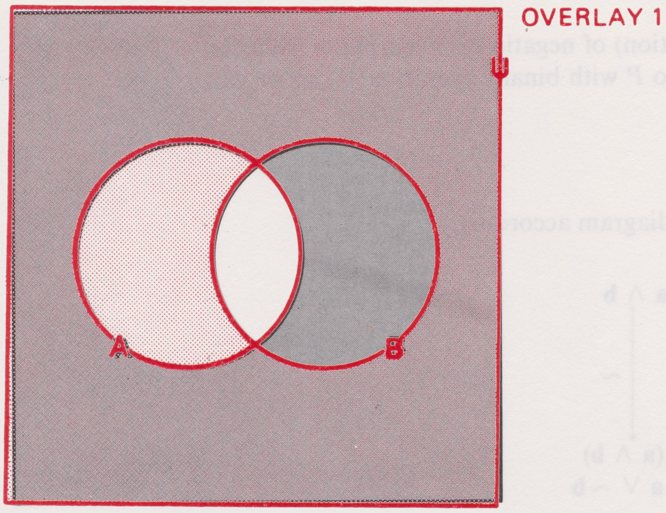
B' shown shaded



$A \cap C$ shaded

(continued from page 13)

In order to solve our problem of finding out what $\sim(\mathbf{a} \wedge \mathbf{b})$ is, we need to express this shaded area, which represents $(A \cap B)'$, in terms of A' and B'.



A' shown shaded

B′ shown shaded

We have drawn these two diagrams* so that they can be superimposed on each other, and you should be able to see that

$$(A \cap B)' = A' \cup B'.$$

That is, the complement of the intersection of A and B is the union of the complements of A and B. We are thus led to infer that

$$\sim(\mathbf{a} \wedge \mathbf{b}) = \sim\mathbf{a} \vee \sim\mathbf{b}$$

and we see that $\square$ is to be interpreted as $\vee$.

We can readily obtain a formal proof of this result by using the truth tables for negation, conjunction and alternation as follows:

a	**b**	$\sim$**a**	$\sim$**b**	**a** $\wedge$ **b**	$\sim$**a** $\vee$ $\sim$**b**	$\sim$(**a** $\wedge$ **b**)
0	0	1	1	0	1	1
0	1	1	0	0	1	1
1	0	0	1	0	1	1
1	1	0	0	1	0	0

* The second diagram is Overlay 1, which you will find in the pocket at the back of this text.

(b)

a	b	c	$b \wedge c$	$a \vee b$	$a \vee c$	$a \vee (b \wedge c)$	$(a \vee b) \wedge (a \vee c)$
0	0	0	0	0	0	0	0
0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

Hence $\mathbf{a} \vee (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \vee \mathbf{b}) \wedge (\mathbf{a} \vee \mathbf{c})$

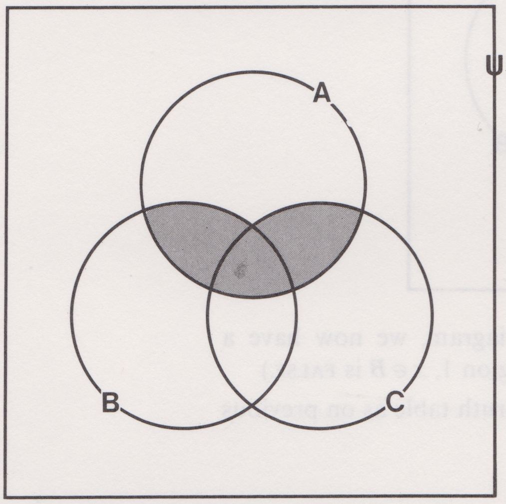
Since $\wedge$ and $\vee$ are commutative, it follows that each of $\wedge$ and $\vee$ is distributive over the other

(ii) $[(\mathbf{a} \wedge \mathbf{b}) \vee (\mathbf{a} \wedge \mathbf{c}) \vee \mathbf{d}] \wedge [(\mathbf{a} \wedge \mathbf{b}) \vee (\mathbf{a} \wedge \mathbf{c}) \vee \mathbf{e}]$

$= ([\mathbf{a} \wedge (\mathbf{b} \vee \mathbf{c})] \vee \mathbf{d}) \wedge ([\mathbf{a} \wedge (\mathbf{b} \vee \mathbf{c})] \vee \mathbf{e})$

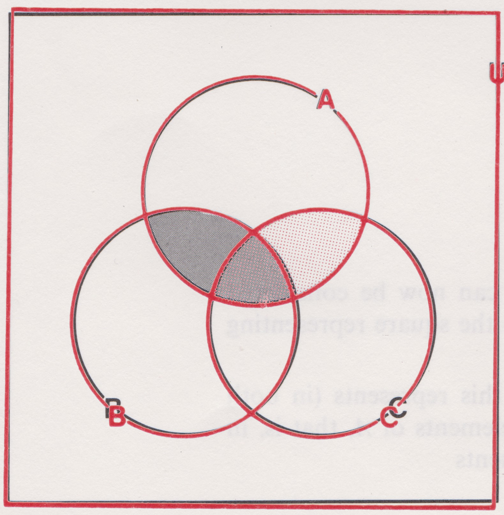
$= [\mathbf{a} \wedge (\mathbf{b} \vee \mathbf{c})] \vee (\mathbf{d} \wedge \mathbf{e})$

(iii) The following diagrams* can be used to verify that intersection is distributive over union:



A ∩ (B ∪ C) shaded

OVERLAY 2



A ∩ B shaded

A ∩ C shaded

* The third diagram is Overlay 2, which you will find in the pocket at the back of this text.